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ABSTRACT

The purpose of this study was to investigate students developing understanding of concepts related to rate of change. Twenty students first participated in a rate of change curriculum unit as part of an after school math and technology program. During the unit, students used motion detectors and related graphing software to create and analyze graphs of changing positions and velocities. They then explored change in a banking context using an interactive diagram called Bank Account. The results of this study suggest that early introduction to rate of change concepts is valuable--elementary students are clearly capable of thinking about and understanding concepts related to rate of change. (Contains 25 references.) (ASK)

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Exploring Rate of Change Through Technology with Elementary Students

ED 441 671

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Rationale:

Why rate of change? Interesting aspects of the domain

The mathematics of change is a topic that students can explore throughout their experiences with school mathematics. Students can approach change on many levels, from elementary arithmetic to advanced calculus, and across many disciplines. In addition, the notion of change is integral to many concepts of more advanced mathematics. A number of researchers advocate an early introduction to the mathematics of change, claiming that understanding of advanced topics such as calculus "develops from basic intuitions that children construct in their daily experiences which physical and symbolic change." (Nemirovsky, 1993) Nemirovsky calls for a long-term view of calculus learning, one that starts much earlier than students' first introduction for formal algebra or geometry. He urges researchers and educators to search for those strands of knowledge that will enable even very young students to deal with calculus-like situations. According to Nemirovsky (1993) and others (Narode, 1986, Tierney & Nemirovsky, 1997), one of these strands of knowledge is the mathematics of change.

Clearly, Nemirovsky (1993) and others are not claiming that by investigating ideas of change and rate of change students are "learning" calculus. Nor do they claim that calculus is easy, or that is should be formally introduced earlier in the curriculum. Instead, they recognize that calculus learning can be facilitated by earlier experiences that allow children to study and represent situations involving change.

Rate of change concepts also have the power to engage students in meaningful and challenging discussions. As students explore rate of change by creating and analyzing qualitative graphs, they encounter results and solutions that are "open to challenging debate and heated discussion." (Narode, 1986) They launch into discussions about why the graphs are

changing, how they are changing, and how their actions and inputs affected the resultant graphs. These dialogues are valuable, for they encourage students to rethink their own understanding, to consider multiple representations, and to look at a problem from various points of view. Not all mathematical activities provide such a rich setting for meaningful discussions.

Using technology to explore rate of change can further enhance students' understanding. As students interact with technology, exploring relationships between movements, input, and changes in graphs, they encounter surprising results, constant feedback and puzzling outcomes, all of which will cause them to wonder. They pursue their own questions, and come to see themselves as capable of asking questions and making sense. In a sense, students' interactions with the concepts and the technology provide a setting for what Duckworth (1996) refers to as "wonderful ideas". According to Duckworth, it is important for teachers to design classroom activities that "suggest wonderful ideas to children- different ideas to different children- as they are caught up in intellectual problems that are real to them" (p. 7). To elicit this sense of wonder and wonderful ideas among students an investigation needs to have elements of inquiry: the students need time and space to ask questions that reflect their personal desires to know and understand, they need to be able to approach questions and problems in ways that make sense to them, and they need to consider how they know something, not just what they know (Nemirovsky, 1993).

What does it mean to understand rate of change in a mathematically powerful way?

While other studies (Narode, 1986, Nemirovsky, 1993, Tierney & Nemirovsky, 1997) have advocated rate of change as an important topic in mathematics, this study attempts to articulate what it means to understand rate of change concepts in a mathematically powerful way. A basic understanding of change and representations of change is needed. A basic understanding of rate of change would include: the ability to construct representations for constant rates of change and changing rates of change, the ability to connect graphs of change with graphs of accumulation, and the ability to analyze graphs of accumulated quantities for the series of changes that produced them, within a given context. But this basic understanding alone is not sufficient. Powerful mathematical understanding entails going beyond a basic conceptual level.

To begin, to achieve a powerful understanding of rate of change, students must be able to analyze and make connections between various contexts involving change. From these experiences exploring change across a variety of contexts, students identify the underlying concepts, such as the relationship between representations of change and representations of accumulated amounts, which are common to multiple situations. As Confrey (2000) states, they begin "to see the like in the unlike." A powerful understanding of rate of change concepts pushes students to what Wilhelm & Confrey (2000) refer to as a "level of abstraction", which allows students to "see" underlying concepts which connect seemingly unrelated representations.

In addition, students who understand rate of change in a powerful way use concepts of change as a "lens" to view and interpret other situations (Confrey, 2000). For example, as students encounter new experiences or representations, they act as "resourceful mathematicians" (Confrey, 1999), using and connecting mathematical ideas, such as ideas about rate of change, as

they make sense of the situation. In this sense, students; knowledge about rate of change becomes "generative" Confrey (1999). According to Confrey, "To be generative, knowledge must be anchored in a range of possible situations for its activation" (p. 14). Students must be able to use knowledge as a lens to interpret and analyze existing models and situations, and also to conceptualize new models.

Finally, students who understand concepts in a mathematically powerful way conjecture, predict, and hypothesize in order to construct a system or structure with consistent rules. As students explore and experiment with models (such as real-time technology that models motion or banking situations), "one must develop a sense of the coherence of the structure, and come to understand and appreciate the interconnections" (Confrey, 1999 p. 15). Working towards an understanding of the structure of a model, and a system of rules that govern the structure , is a critical element to powerful mathematical understanding. "Structure is essential to emphasize within any mathematical activity. ... Without a discussion of structure, one could never move from purpose and description to explanation and argument." (Confrey, 1999, p. 10) Models and micro-worlds, such as Bank Account (Confrey & Maloney, 1998), that represent rate of change concepts, allow students to experiment and receive immediate feedback as they attempt to make sense of a system and the rules that govern its behavior. As students use Bank Account, they expect it to operate by a consistent, predictable system of rules, and they experiment to investigate these rules. Students' abilities to conjecture, hypothesize and construct this system of rules is an important element of powerful mathematical understanding.

Exploring rate of change in an urban, predominantly minority setting.

Not only does this study attempt to illustrate what a powerful understanding of rate of change entails, it also attempts to document how this understanding develops among students in an urban, low SES, predominantly minority elementary school, students whose abilities and intellect are often overlooked. For example, research has repeatedly illustrated that our educational system is not equally effective for all students, regardless of their race, ethnicity, gender or social class. The comparably lower achievement of Hispanic students, particularly those acquiring English as a second language has been demonstrated by numerous assessment measures. (Dossey et. al, 1988, Secada, 1992, Tate, 1997). The general picture presented by the NAEP and other standardized assessments is that whites perform better than Hispanic and African American students, and that these disparities increase over time. This low performance among some minority groups becomes interpreted as a lack of intelligence, or lack of "readiness skills" needed to succeed in school mathematics, rather than a failure of the school to meet the needs of the students.(Mesa, 1998). The insights and understandings that students developed over the course of this study demonstrate that students who are typically considered "low achieving" are quite capable of powerful mathematical thinking.

In addition, this study attempts to document how students in an urban, low SES, predominantly minority school use technology as a tool to make sense of rate of change concepts. Research has documented that while there has been an influx of computers into schools, all schools have not benefited equally. Urban school districts with a higher proportion of minority and low SES students generally have a lower computer-to- student ratio than wealthier districts. (U.S. Congress, 1995, Besser, 1993) Availability of technology in classrooms is not the only issue to consider. Researchers have also investigated the kinds of

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opportunities students have to use technology in school. A study conducted by the Policy Information Center of the Educational Testing Service (Wenglinsky, 1998) found that students who are poor, urban, and minority were less likely to participate in higher-level computer uses than non-poor and suburban students. The previous group most often engaged in low-level computer uses, such as drill and practice activities. (Wenglinsky, 1996, Besser, 1993) This study demonstrates that minority students can and do engage in higher level uses of technology, using computer technology as a tool for thinking, experimenting and conjecturing, and that such higher level use of technology actually promote a powerful understanding of rate of change concepts.

Review of the Literature: Definition of relevant terms

•Mathematics of change

The mathematics of change refers to mathematical concepts that are related to both physical and symbolic change. Physical change can refer to changes in motion of objects, seasonal changes, changes due to growth, and changes in the flow of air or water. Symbolic change involves increases and decreases in numbers, graphs, or shapes. As change is central to all of our experience, the mathematics of change becomes a strand of mathematical concepts that students can explore on many levels, from primary grade students exploring graphs that depict changes in motion to college students exploring calculus theorems based on the notion of change.

•Rate of change

According to some mathematics educators, a rate is an intensive quantity, a relationship between one quantity and one unit of another quantity. (Kaput & West, 1994) Common rates

include speeds (meters per second, miles per hour), and prices. Confrey & Smith (1996) advocate a different view of rate. A rate is not limited to relationships between unlike quantities, as Kaput and West argue. For example, inflation, expressed as a relationship between dollars to dollars, is clearly a rate. According to Confrey and Smith, a rate is a ratio that can be quantitatively increased or decreased. Quantities can involve a constant rate of change, meaning that the relationship between the two quantities does not change. (E.g. A car travels at a constant rate of 35 miles per hour) A changing rate of change is also possible. (E.g. A car starts out at 0 miles per hour, and then gradually speeds up to a rate of 60 miles per hour) Within the context of this paper, I will refer to speed as a rate of change, constant speed as a constant rate of change, and increasing or decreasing speeds as changing rates of change.

•Discrete change versus continuous change

Discrete change refers to changes in quantities that can be measured by discrete amounts, usually with whole numbers. ($50.00 in the bank versus $25.00 in the bank). Continuous change describes changes in quantities that are not measured in discrete, whole number amounts (gradually increasing speed, gradually decreasing temperature). (Nemirovsky et. al., 1998)

•Quantity (how much?) versus change in quantities (how much has it changed by?)

A quantity refers to an accumulated amount. For example, how much money is in the bank now (balance)? How far has the car traveled so far? A change in quantity refers to sequences of changes that affect the accumulated quantity. For instance, how much was withdrawn or deposited into the bank (transactions)? How much further or closer has the car travelled? While distinguishing these two measures is important, what is more important is encouraging students to make connections between them. One of the "big ideas" of the mathematics of change is understanding how sequences of changes affect an accumulated amount.

•Motion detectors

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Motion detectors are computer based data probe devices that measure and display the position, velocity, and acceleration of an object over time. The motion detectors can accurately detect objects from 0.5 meters to 6 meters away. The data are displayed in graphical form on the computer screen as the measurements are taken. The software usually creates two kinds of graphs, a position versus time graph and a velocity versus time graph. Students can save graphs to make comparisons later, or display multiple graphs on the screen at one time.

Students' alternative conceptions about representations of change

Various studies have documented the difficulties that many students experience in representing and interpreting representations of rate of change. MacFarlane, A., Friedler, Y., Warwick, P., & Chaplain, R. (1995) et. al. found that students in a control group who experienced a traditional approach to recording data (changes in temperature over time) and then manually plotting the data on a grid, showed no improvement in their understanding of graphs as representations of changes in variables. (MacFarlane et. al.) This lack of improvement was despite the fact that these students were taught by experienced, highly skilled teachers. Mokros and Tinker (1987) explain that students have a strong tendency to view graphs of change as pictures, instead of as symbolic representations. They rely on existing knowledge of physical phenomenon (such as motion) and superimpose this knowledge on the graphical representation. For example, students examining a velocity versus time graph of increasing and then decreasing velocity might interpret the graph to represent the motion of an object that travels up and then down a hill.

These alternative conceptions about representations of change may be due to how teachers often present the concepts to students. For example, in traditional curricula, children are asked to draw graphs representing changes before they ever have a chance to make sense of the

graphs. (MacFarlane et. al., 1995) They spend considerable amounts of time plotting points on a coordinate grid, and have few opportunities to actually think conceptually about what they are graphing (What is changing? How is it changing? How do you go from bar graphs to coordinate pairs?)

Instead, researchers suggest that students should first explore representations of change more holistically and dynamically. They should have opportunities to manipulate physical variables (change the speed or direction of their movement, change the amount of input) and immediately observe the results. Relying on technology to dynamically create and adjust the graphs allows students to spend their time thinking about how the variables interact and how the changes in input effect the shape of the graph. In addition, students should have opportunities to use technology to construct multiple representations of change, and to investigate the relationships between these representations. Even young children, with little or no prior knowledge of coordinate graphing, can successfully grasp concepts of change and rates of change when they use technology as a tool to explore graphs conceptually. (Narode, 1986)

What students do understand about change and representations of change

While awareness of students' alternate conceptions may help teachers more carefully design instruction and better prepare for possible points of confusion, focusing only on students' "misconceptions" is not sufficient. A misconceptions approach may lead to seeing students for their deficits that need to be overcome, instead of for their strengths and competencies. Thus it is critical to document what students do understand about representing and interpreting rate of change. For this to occur, researchers need to find a different way of measuring and "seeing" students' understanding. Nemirovsky et. al. (1998) advocate an approach that looks at students as they invent their own representational systems, paying attention to their "conversations, their

emerging resources, and how they relate to the use of tools." (p. 123) Research has documented that as students create and interpret representations of change, they become aware of the meaning of the graph, and the features that are relevant and important. (Nemirovsky et. al., 1998) When these representations are linked to real physical phenomenon, such as motion, which students can relate to their experiences in the world, the knowledge that emerges may be even more developed and interconnected. (TERC, no date) Thus observing students as they construct and analyze representations of change may provide a rich source of data about students' understanding.

Based on a case study of two students as they used motion detectors and graphing software, Nemirovsky et. al. (1998) developed a framework for what it means for students to learn to interpret rate of change representations. According to their research, children construct a graphical space; a lived-in space that they create based on their experiences. As they intentionally explore movement within this space, it becomes part of their discussions and gestures. Students also develop a perspective of the tools there are using. In this case, the students had access to motion detectors, computer software, and various objects that could move (trains, buttons with sensors, and their bodies). They developed a sense of what was possible, impossible, and likely using the tools. The third component of the framework for understanding how students learn to interpret representations of change was fusion. The idea of fusion is that children merge the qualities of the graph with the events that they represent. The graph and the events cease to be distinct entities in their minds; instead, they see the graph as a shape and the graph as a response to their actions as one in the same.

Students' actions as they come to understand representations of change

As students make sense of representations of change, their actions are intentional. They move, adjust, and create in an effort to accomplish something (creating a particular graph) or understand something (how moving faster affects the graph). (Nemirovsky, 1998) Students also pay close attention to relationships. They make connections between measures, such as that a decrease in the height of the graph is related to a closer position to the motion detector. Students also form relationships in their minds between different representations of motion that they experience during the investigation. (Noble et. al., no date) Without instruction from the teacher, students figure out which aspects of creating the graphs are significant, and which are idiosyncratic. They realize that their position and their speed have a direct impact on the shape of the graph, while other factors, such as how they hold a sensor, may influence how well the detector picks up their changes in motion. As they continually attempt to construct meaning of the representations of change, some students may even place themselves within the context of the graph. For example, two 5th grade students came to understand graphs of motion by creating motion stories and placing themselves within those stories. (Noble et. al., no date)

Students' understanding of changes and accumulated amounts

While various studies (Noble et. al., Nemirovsky et. al., 1998, MacFarlane et. al., 1995, Mokros et. al., 1987) have documented how students construct meaning of representations of change, few studies have explored students' understandings of the differences between changes and accumulated amounts. In a study of elementary school children, Nemirovsky (1993) found that children were able to solve problems in which the initial and final quantities were given, and they had to generate possible sequences of changes. (You start with 4 blocks in the bag. Can you make 2 (or any given number) changes so that you end up with 7 blocks in the bag?) Students realized that you could decompose numbers to create more changes, without effecting

the net change. (Adding 1 and 1 and 1 in three changes has the same result as adding 3 in one change.) Students also discovered that the reverse was true; they could decrease the number of changes, without affecting the net change, by composing the numbers. Surprisingly, the students also demonstrated an amazing capacity to keep track of both changes in quantities and accumulated amounts, and to realize the differences between the two measures. They understood that a number could adequately represent an accumulated quantity, but not a change. Since changes were directional (adding in or taking away) they needed to indicate both the direction and the amount of the change.

Students are also able to spontaneously create graphs, pictures, and/or charts that represent changing situations. (Tierney & Nemirovsky, 1991, Nemirovsky, 1993) Often, the students representations are data-driven, that is the main goal of the representation was to communicate all information about the situation. Their representations preserved the important information (about changes and accumulated amounts) and communicated the information in a way that others could understand. The students were not as concerned about presenting the information systematically, using well defined sets of symbols, equal intervals, or creating graphs that could encompass all possible options. While this study is certainly promising, demonstrating that children as young as first grade can successfully make sense of and represent changes and accumulate quantities, more research is needed in this area.

Wilhelm & Confrey (1998, 2000) conducted a series of clinical interviews with high school students and found that the use of a computer-based Interactive Diagram © (Confrey & Maloney, 1998) helped deepen students' understandings of rate of change and accumulation. Students in this study successfully used the Bank Account Interactive Diagram to predict, model, and check their ideas about changes (transactions in this context) and accumulations (balances in this context). Wilhelm et. al. argues that the Bank Account Interactive Diagram © (Confrey &

Maloney, 1998) served to bridge the gap between qualitative and numerical reasoning as students worked with the language of steps, steepness, and slope. The technology provided an effective tool that students used to make connections between continuous graphs and the concept of slope or "steepness" as a numerical ratio. Clearly, more studies need to explore how technology can facilitate students' abilities to create and interpret graphs that involve rate of change.

Role of previous knowledge and experience

As students try to make sense of representations of change, they do draw upon prior knowledge and experiences. Carraher et. al. (1995) found that students repeatedly try to find connections to situations they are familiar with. For example, in trying to understand how the motion detector worked, one student created an analogy between motion detectors and what she knew about sonar mapping of the ocean floor. "Sort of like, you're scanning something, you can tell how tall it is. ... I remember seeing a TV program they did this to study the ocean, ... and on the computer it picked out, sort of landscape on the bottom of the water. ... So it's almost as if you're marking the bottom of the ocean and it's showing what it is on the screen." (p. 4) Students also relied upon their knowledge of numbers to talk about and compare graphs representing different speeds. They were able to use the properties of a system they already knew (the number system) to make sense of a new situation, graphs of varying speeds. Students analyzed the graphs for variations in speed, and then assigned numerical values to each graph, allowing them to compare one speed to another. This process of using numbers to represent the speed of various movements seemed to facilitate the process of understanding speed for some students. In this study, students did not formally "transfer" existing knowledge to novel situations. Instead, they engaged in "a dynamic and open ended way of relating previous experiences to new ones." (p. 20) Their understanding of new situations was enriched by their

previous knowledge, and their previous knowledge was expanded and enriched by their new experiences.

Role of technology

Numerous studies have documented the positive influence of technology in facilitating understanding of rate of change. MacFarlane et. al. (1995) studied the impact of using computer-based laboratories and accompanying curricula in elementary school classrooms. They found that using data probes and graphing software greatly enhanced students' ability to read and interpret graphs of change in a variable over time. The technology allowed the students to manipulate physical variables and observe the results, which may have helped them understand the connections between change in variable and change in the graph. Students' ability to create graphs to represent a given situation involving change also improved.

Mokros and Tinker (1987) found that using microcomputer-based laboratories to explore changes in motion effectively "removed" students' misconceptions about graphical representations of change. They students did not consider the graph to be a "picture" of the movement, but a symbolic representation of changes in position and velocity over time. They suggest four possible reasons for the effectiveness of the computer-based laboratories: "MBL uses multiple modalities; it pairs, in real time, events with their symbolic representations; it provides genuine scientific experiences; and it eliminates the drudgery of graph production." (p. 381) The immediate, accurate feedback provided by the technology may have helped students develop a deeper understanding of the graphs. Mokros & Tinker also emphasize that computer-based devices provide students with a real-time link between a concrete, physical experience and the symbolic representation of that experience. Other studies also stress the value of technology

in allowing students to explore physical embodiments (motion) of abstract mathematical concepts (rate of change). (TERC, no date)

Technology can also promote learning as it becomes a tool that creates a "problematic" for students to solve. (Confrey, 1991, Nemirovsky et. al., 1998) Initially, the problematic that the technology presents is simple: figure out how the this tool (the motion detector) works. What does it detect? What kinds of motion provoke salient responses in the graph? As students progress and discover more about the tool, the problematic becomes more complex: find ways to create a given graph. Is this graph possible? What happens if you add relative motion (move your body and move the motion detector)? These "problematics" will emerge continuously as students interact with the technology and reflect upon the results.

Given young children's competencies in interpreting and creating representations of change, and the potential benefits of using technology to enhance these understandings, this study further explored how elementary students make sense of rate of change concepts.

The "Rate of Change" study

The purpose of this study was to investigate students' developing understanding of concepts related to rate of change. Students first participated in a rate of change curriculum unit as part of an after school math and technology program. During the unit, they used motion detectors and related graphing software to create and analyze graphs of changing positions and velocities. Students then explored change in a banking context, using an interactive diagram called Bank Account © (Confrey & Maloney, 1998). Bank Account involves a graph of daily transactions (changes) and a balance graph that displays accumulated quantities. Students can manipulate the transaction graph by depositing or withdrawing money, and the Bank Account program creates the accompanying balance graph. The program allows students to control whether or not the balance graph is displayed, so they can first predict the shape of the balance graph and then display the balance graph to test their prediction. (see figure A)

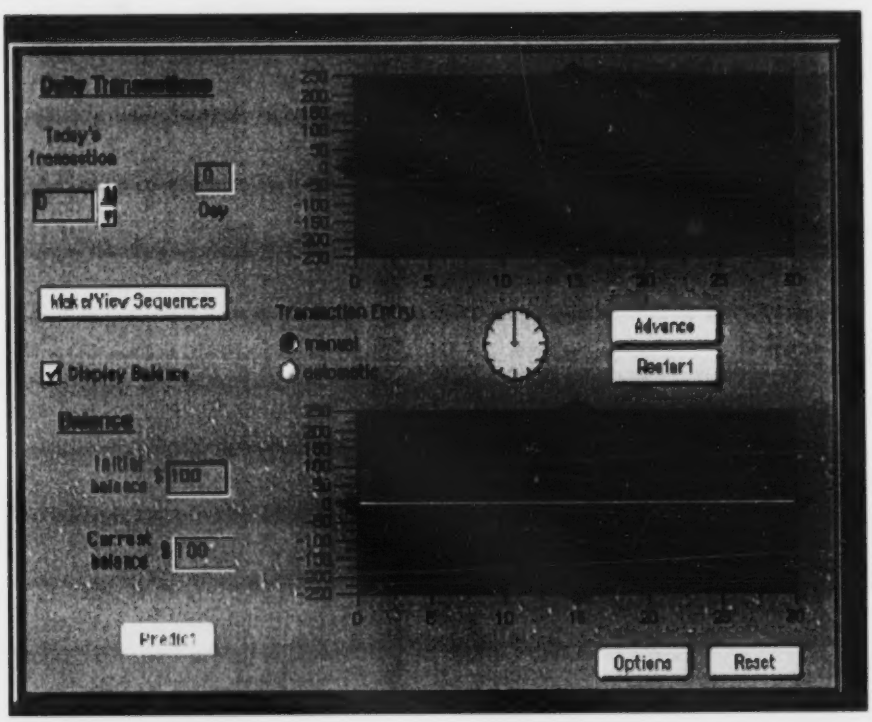

Figure A

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Upon completion of the unit, student understanding was assessed through in-depth clinical interviews. The study investigated how studying the same concepts in different contexts helped students reach a deeper level of understanding, and how technology impacted their understanding.

The general questions that guided the study were:

1. To what extent did students develop a *basic understanding* of rate of change concepts?

   For example, how do students make sense of the relationships between graphs of change and the corresponding graphs of accumulation (bank account), and graphs of position vs. time and the corresponding graphs of velocity vs. time (motion detectors)?

2. To what extent did students develop a *powerful mathematical understanding* of rate of change concepts? For example:

   a. Are students able to make connections between representations of rate of change from different contexts, to "see the like in the unlike"? (Confrey, 2000) How do those connections impact their overall understanding of rate of change concepts? To what extent are students able to connect qualitative graphs of distance vs. time and velocity vs. time created using motion graphs of transactions and accumulated amounts using Bank Account interactive diagram?

   b. To what extent are students able to use their understanding of rate of change as a "lens" to interpret and analyze other situations and representations?

   c. As students explore models of rate of change concepts (i.e. Bank Account), how to they conjecture and reason to make sense of the model as a system governed by consistent rules and principles?

3. And finally, how does technology (motion detectors, graphing software, interactive diagrams) seem to influence/facilitate students' understanding?

Methods

This section will describe the project setting, study participants, curriculum intervention, clinical interviews, and the methods of data collection and analysis.

Project Setting:

This study was conducted from January to March , 2000, at Longview Elementary School (a pseudonym), a preschool through fifth grade campus that is part of the Austin Independent School District in Austin, Texas. Longview School has approximately 640 students, 76% Hispanic, 9% African American, 15% Caucasian and other racial and ethnic groups. 80 % of the students at Longview are eligible for the district's free or reduced lunch program. Longview Elementary School, which is part of the Tree High School feeder cluster, is involved in a systemic research initiative conducted by researchers at the University of Texas at Austin. This year, researchers from the University began to work with third, fourth, and fifth grade teachers at Longview.. This study was designed to introduce several new technologies to students and teachers at the school, technologies which 4th and 5th grade teachers will explore in depth during summer workshops sponsored by the University.

The study began with an after school program conducted in two different locations on the Longview School campus. For the first three sessions, students met in the school's intermediate computer lab, which was equipped with 18-20 Power Macintosh computers. Students worked cooperatively in small groups of 3 to 4, sharing computers and motion detector equipment. For the last three sessions, we met in one of the fifth grade teacher's classrooms. The Bank Account software we needed to use did not run on the computers in the lab, and since the fifth grade classrooms all had mini-labs of iMac computers, we decided to work there. The students again

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worked in small groups, sharing the 4 computers in one classroom and 2 addition computers in an adjoining room.

Study Participants:

A total of 20 fifth grade students and one fifth grade teacher participated in the after school program. Several months before the program began, I met with the three fifth grade teachers at Longview School to discuss the program with them. I reviewed the curriculum, briefly describing the activities and the concepts that the students would explore during each session. Together, we decided that 6 to 7 students from each classroom could participate. We arranged district transportation to take students home each week, so that more students might be able to attend. While I allowed each teacher to determine which students would participate from his/her classroom, I explained that I wanted a range of students, both in terms of ability and interest in mathematics. Each teacher later assured me that the final group represented a wide variety of students. One of the fifth grade teachers also volunteered to attend the sessions because he was interested in learning more about the technology we were using.

Curriculum Intervention:

The after school program consisted of a six-week curriculum unit focused on rate of change concepts. The students met every Wednesday, and each session lasted one and a half hours.. What follows is a brief overview of the 6 sessions.

Session 1: Introduction to qualitative graphing via motion detector

During the first session, students worked in small groups to figure out how to interact with the motion detectors to produce graphs that reflected their movement. Students first created their own graphs, and then attempted to make several target graphs that I presented to them. At the end of the session, I facilitated a discussion around the following questions: How does the

motion detector work? What kinds of graphs were you able to make, and how did you make them? What did you discover about high and low portions of the graph?

Sessions 2 and 3: Relationships between distance, velocity and time using motion detectors

In the following sessions, students continued to use motion detectors to create and interpret graphs of changing position and velocity. I provided each group with a collection of graphs that they needed to construct. (See appendix A and B) These graphs were carefully chosen to help the students attend to significant aspects of motion: variation in speed (slow vs. fast), changing speeds versus constant speeds, forward versus backward motion, and changing position versus constant position. As students figured out how to make the graphs, they made predictions and then tested those predictions by enacting the movements.

They explored two different graphs representing motion during these sessions: graphs of position versus time and graphs of velocity versus time. The software program created both graphs simultaneously, allowing students to compare the position and velocity graphs for a single motion. While students were working, I posed questions to them such as: Why did the position graph go up? What do you notice about the velocity graph? How does the velocity graph represent his movement? What would happen if another students walked backward again, but this time faster?

Session 4: Exploration of sequences of changes and amounts using In and Out of Bags context (Nemirovsky, 1993, Tierney & Nemirovsky, 1997)

During this session, students investigated sequences of changes in a different context that allowed them to explore discrete changes instead of continuous changes. (Nemirovsky, 1993) They worked on a variety of problems related to this context such as: creating graphs to represent a sequence of changes (adding and taking away) of cubes in a bag and to represent the total amount of cubes in a bag. The purpose of these activities was to help students develop some

general ideas about the interplay between changes and accumulated amounts. By the end of the session, most students were able to represent sequences of changes and the corresponding accumulated amounts, and talk about the relationships between the measures.

Sessions 5 and 6: Exploration of relationships between representations of changes and representations of accumulated quantity using Bank Account Interactive Diagram © (Confrey & Maloney, 1998)

In these sessions, students worked with the Bank Account Interactive Diagram (ID) to explore connections between graphs of changes (daily transactions) and graphs of accumulated quantities (balance graphs). Bank Account is designed to assist students in developing intuitive ideas about these relationships. At first, students worked to understand the impact of a variety of different patterns of transactions on the shape of the balance graph. Once they understood the process in a forward direction (manipulating transactions to produce a balance graph) they investigated how to create various balance graphs (that are given) by specifying a sequence of transactions.

Clinical Interviews:

When the after school program finished, I selected six students to participate in clinical interviews. While I interacted with many students during our weekly sessions, I wanted the opportunity to further probe their thinking so that I might better understand how the students made sense of rate of change concepts. Time constraints did not permit me to interview all of the students, so I chose six students who attended and actively participated in all of the after school sessions. I also intentionally chose two students from each of the three fifth grade classrooms. The six students included two Caucasian girls, Kristina and Heather, two Hispanic girls, Fatima and Carmen, one Hispanic boy, Carlos, and one African American boy, James.

One of the Hispanic students, Fatima, was a dominant Spanish speaker who completed the interview in Spanish. Each interview lasted approximately one hour and 15 minutes.

Comments about each of the clinical interview participants:

Kristina, a Caucasian girl, was very eager to participate in the clinical interview. She was a leader in the after school program, frequently helping other students and volunteering to share her strategies with the whole group. When she did not agree with another students' prediction or results, Kristina often challenged the student and justified her own position. Kristina completed the interview in two sessions, both after school.

Carlos, a Hispanic boy, was an active participant in all six of the after school sessions. He made frequent comments about how much he enjoyed working with the technology. Carlos had to leave several of the sessions a few minutes early, and he was always anxious to find out what he had missed. Carlos completed the interview in two sessions after school.

Fatima, a Hispanic girl, is a native Spanish speaker who just moved to the United States from Mexico the summer before she began 5th grade. While she is beginning to understand a fair amount of English, Fatima is most comfortable interacting in Spanish. All of her participation in the after school program, and the entire clinical interview, was conducted in Spanish. Fatima completed the interview in one session during the school day. In the analysis section of the paper, I include some of Fatima's comments, both as she stated them in Spanish, and then translated into English.

James, an African American boy, acted almost like an assistant to me during this project. He demonstrated a genuine interest in the technology, and often came early to help me set up the motion detectors and prepare the computers for use. He continually asked questions about how the Bank Account applet worked, how the motion detector really detected motion, and thought of

ways that he wanted to adapt the technology to meet his needs. James completed the interview in two sessions, both during the school day.

Heather, a Caucasian girl, participated rather quietly in the after school sessions. She frequently worked with Carmen or Kristina, usually allowing them to talk and share the groups' ideas with the class. Heather took the activities and projects very seriously, and often spent extra time on the activities. Heather completed the interview in one session, during the school day.

Carmen, a Hispanic girl, frequently worked with Fatima during the after school program. As Carmen was proficient in both Spanish and English, she often translated instructions and comments for Fatima when she did not understand. Carmen completed the interview in one session, during the school day.

Clinical Interview Questions:

The interview (see appendix C) consisted of a variety of questions designed to probe students' thinking about rate of change concepts. The interview questions were constructed by Wilhelm (1998) and used in her study to explore high school students' thinking about rate of change concepts. I made minor adaptations to Wilhelm's interview, such as deleting several questions due to time limitations, and changing the wording of questions to make them more understandable for younger students. The first portion of the interview asked students to analyze and create graphs representing motion, including graphs of position versus time and velocity versus time. Students used the motion detector probes and the accompanying graphing software during the interview. The second section explored the relationship between graphs of change and graphs of accumulation using the Bank Account Interactive Diagram. The second portion of the interview presented the students with several banking situations which they had to discuss

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and represent using the software. The final section urged students to think about the possible connections between the various rate of change contexts that they have explored.

Data Collection and Analysis Methods

During the six after school sessions, I collected a variety of forms of data. All sessions were videotaped, focusing on small groups of children as they worked, and on my interactions with those children. I also collected samples of student work whenever appropriate. The work samples included: graphs the students created, written predictions and explanations, drawings, and reflections they wrote after an activity. All clinical interview sessions were videotaped, and the work students produced during the interview, such as graphs and predictions, was collected.

While all of the data provided rich sources of information about the students' understanding, I chose to focus on the clinical interview tapes for the purposes of this paper. As I analyzed the data, I used data analysis techniques that resemble the process of analytic induction. To begin, all interviews were transcribed, because I felt that transcribing the students' comments would provide me with a sense of the richness and complexity of their understanding. I then reviewed the transcripts of the six interviews, making notes about insights, ideas, and trends that I noticed in the data. This initial analysis focused on the data as a whole, not analyzing the transcripts line by line. The data was then reviewed again, and I made more specific notes about individual students' responses to certain questions. This time, my focus was on questions that were part of the second and third sections of the interview, questions that addressed students' understandings of rate of change in a banking context and students' abilities to make connections across contexts. I found that these questions elicited a wider variety of responses and provided me with more information about students' understandings of different

representations of change. I then reviewed my notes, looking for patterns, trends, or insights that emerged across the data. I developed a list of those patterns (which are now more like assertions), and looked back at the data for evidence of each assertion. Finally, I searched for connections among the patterns and trends that I noticed.

Results

I was struck by the richness and depth of the students' thinking about rate of change as they solved problems during the clinical interviews. While the students' comments and responses were varied, after reviewing the tapes several times, I was able to find patterns and commonalties in their understandings. I will discuss each pattern or "trend" in the data, and then provide transcribed excerpts from the videotaped interviews to illustrate the nature of the students' understanding.

1. **Students developed a basic understanding of different representations of change within the banking context**

Several questions in the clinical interview were designed to assess students' basic understanding of different representations of change and the connections between those representations within a single context. For example, during the Bank Account portion of the interview, students were presented with a series of constant transactions (adding 50 dollars per day to a bank account) and asked to create both a graph of change (transaction graph) and a graph of accumulation (balance graph) to represent the situation. Once students created the graphs, I probed their thinking to determine whether they understand the connection between the two representations. A similar question assessed students' understanding of changing rates of

change. Presented with a transaction graph depicting a changing rate of change (see figure B) students were asked to construct the corresponding accumulation, or balance graph.

Other questions asked students to analyze balance graphs for the series of transactions, or changes, that produced them. Finally, throughout this portion of the interview, I asked students

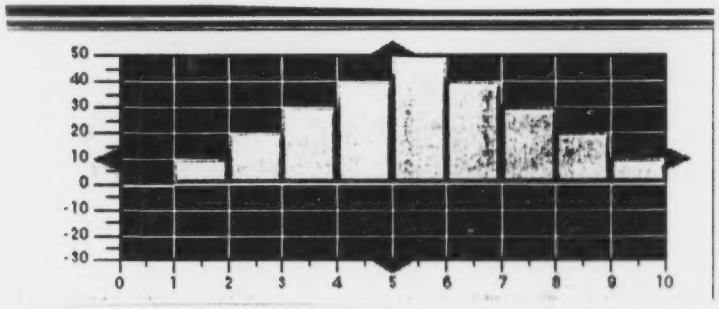

Figure B

to think about the differences between constant rates of change and changing rates of change, to determine whether they could distinguish between the two representations. The following table displays students' performance on the questions described above. (Yes means that students successfully answered the question and demonstrated understanding of the underlying concept)

Student	**Construct representations for a constant rate of change**	**Construct a balance graph given a changing rate of change**	**Analyze balance graphs for the series of changes that produced them**	**Distinguish between constant and changing rates of change**
Kristina	YES	YES	YES	YES
James	YES	YES	YES	YES
Carmen	YES	YES, though initial prediction was incorrect	YES	YES
Fatima	YES	YES	YES	YES
Heather	YES	YES, after first producing an incorrect representation	YES	YES
Carlos	YES	YES, though initial prediction was incorrect	YES	YES

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As the table illustrates, all six students demonstrated basic conceptual understanding of different representations of change within the banking context. Given a series of transactions representing a constant rate of change, all students constructed graphs of change and accumulation to model the situation, and made connections between the two graphs. For example, in response to my question, "How does the top graph (transaction graph) relate to the bottom graph (balance graph)?" students made the following comments.

> James: The top graph, it adds 50 each day, and it only display that, you're adding 50, 50, 50, 50. The balance graph is going to show how much you have all together. Each day it adds on to your initial 100. ... Right here (points to step on balance graph) it show 50 up. Each time it has 50 more.
>
> Kristina: It (balance graph) goes up the same amount each time. Each day, you are adding 50 there (points to the transaction graph) and you are adding 50 there (points to the balance graph)"
>
> Fatima: En esta (transacciones) es donde te vas sumando o te vas quitando. Entonces, un día pones 100, otro día pones otro 100. Y la otra (balance graph) te vas sumando, y te va, como agregando la cantidad, y te dice cuanto hay en total.
> *Translation:*
> *Fatima: This one (transaction graph) is where you are adding or taking away. So, one day you put in 100, another day, another 100. And the other one (balance graph), you go on adding, and it goes, it like accumulates the amount, and it tells you how much there is in all.*

Students were also able to interpret a transaction graph representing a changing rate of change (figure B), although these interpretations were more difficult for some students. All students correctly analyzed the first portion of the graph, concluding that they needed to add increasing ammounts to their bank account each day. As Kristina stated, "Ok, you start at zero, then you add 10, then add 20 so you would be at 30 (points to balance graph). Then you are adding 30, so you would be at 60. Then you're adding 40, so you'd be at 90. Then you're adding 50, so you'd be at 140." However, when the students interpreted the second portion of the graph which represented a decreasing, yet still positive rate of change, they were not as confident in their responses. Three of the six students initially interpreted this decrease to signify

taking money out of the bank account. For example, after correctly constructing the first part of the balance graph, representing an increasing rate of change, Carlos made the following comment as he began to work on the second part of the graph.

> Carlos: Then you subtract 10 every day, subtract 10, subtract 10 then another 10, so it would show your balance going down, show as much as you have.
> Erin: How did you know that you would subtract ten every day?
> Carlos: Cause it shows, ok … (pause) Man, I did it wrong. It shows you added less money, not as much money. You would still be going up though, it would keep going up. (points to the balance graph)

Both Carmen and Heather made similar mistakes initially, and then reasoned through the problem as Carlos did to arrive at a correct interpretation.

All students were also able to analyze a collection of three balance graphs and determine the sequence of changes (transactions) that produced those graphs. In addition to correctly creating the transaction graphs, students formed connections between the three balance graphs, realizing that they all depicted constant rates of change. For example, in response to my question "What do you notice about these three balance graphs, how are they alike and how are they different?", students made the following remarks.

> James: They're getting, going up the same amount on each one. They're not adding 10, then 20, then 50. This one is adding 20 each day, this one is adding 100 each day, and this one's adding 50 each day. The same amount each day.

> Fatima: Pues, de que cada día vas subiendo cada vez lo mismo. Esta, sumas 100, 100. El otro tiene 50, 100, 150, lo vas sumando cada vez 50. .. Como que le vas sumando no los mismos números, pero siempre lo mismo lo mismo, lo mismo.
> *Translation: Well, each day you are always going up the same. In this one, you add 100, 100. The other one has 50, 100, 150, you are adding 50 every time. … It is like you are adding, not the same numbers, but always the same, the same, the same.*

Finally, throughout the interview, students demonstrated that they saw a clear distinction between graph representing a constant rate of change and graphs representing a changing rate of change. They used language and terminology they constructed themselves to explain this

distinction. For example, students used the term "staircase" to refer to balance graphs representing constant rates of change. (This term was first used by several students during the 5th session of the after school program, and became part of our common language for talking about graphs.) A "staircase" graph referred to any graph that increased or decreased by a constant amount over time, a graph depicting a constant rate of change. Fatima's comment below illustrates her notion of the meaning of "staircase."

> E: Y porque piensas que tendría la forma de una escalera?
> F: Si, porque su sumabas 100 y luego 200, no tendría la forma de una escalera, pero si sumas 100, y luego 100 100 100 siempre, tiene la forma de una escalera porque cada vez vas sumando lo mismo.
> *Translation:*
> *E: And why do you think it would have the shape of a staircase?*
> *F: Yes, because if you add 100, and then 200, it would not have the shape of a staircase. But if you add 100, then 100 100 100, always the same, it has the shape of a staircase because you are always adding the same amount.*

Other students also articulated this idea that certain graphs which depict a constant rate of change can be described as staircase graphs. Heather's comment demonstrates that she also paid attention to the transaction graph, and noticed that a constant rate of change produces a flat transaction graph.

> H: Cause you are adding 50 every day ..this top graph would be flat. And the bottom graph, would look like you are adding 50, adding 50, adding 50. Like a staircase.

> K: It's a staircase graph. It's going up the same amount each time, and stairs go up the same amount each time. It's not adding like 10, 20, 30 .. that would be like stairs that stepped little, big, real big.. and that wouldn't really be stairs.

Not only did students recognize a constant rate of change, but they differentiated between representations of a constant and a changing rate of change. For example, many students began talking about graphs with an increasing, positive rate of change as staircases that were not normal staircases, but staircases that were curved. Carmen distinguished between "straight staircases" (graphs of a constant rate of change) and "curvy staircases" (balance graphs with a

changing rate of change). What follows are comments from Heather and Fatima and they make sense of the differences between the different rates of change (constant versus changing).

> H: It looks weird, kind of like stairs, but different. .. Actually, it goes like this ,curved. (traces out a curve shape in the air) Add a little, add a little more, add a little more than you did the last time, add a little more than you did the last time, and then it stops.
>
> F: Tiene como escalera, no más que no es. No son escaleras escaleras, son escaleras en curva. ..Es diferente, porque aquí se suben dos escalones, aquí se le suben tres, aquí 2, aquí se le suben uno, por eso. Porque cada vez le vas sumando más y luego sumando menos.
> *Translation:*
> *F: It is like a staircase, but it isn't. It is not a staircase staircase, it is a staircase that curves. Its different, because here you go up two steps, and here you go up three, and here 2, and here you go up one, that's why. Because every time you are adding more, and then adding less.*

The above examples demonstrate that the students developed a solid basic understanding of different representations of change and the connections between those representations within a single context. As I reviewed the data, it was also clear that the students' understanding went beyond this basic, conceptual level. In fact, the students' responses demonstrated each of the elements of a powerful mathematical understanding of rate of change: 1. The ability to make connections across contexts. 2. The ability to use understanding as a lens to interpret new situations, and 3. The ability to conjecture and experiment to build a system of consistent rules. The three sections that follow illustrate how students demonstrated each of these components of a powerful mathematical understanding of rate of change.

2. Students were able to make some connections between representations of change across contexts, demonstrating a powerful mathematical understanding of rate of change.

One of the questions that guided this study was: To what extent are students able to make connections between different representations of rate of change, to "see the like in the unlike"? In a similar study, Wilhelm and Confrey (1998, 2000) found that high school students were able to relate their experiences with motion detectors and bank account. Some students even reached a level of abstraction that allowed them to connect the graphs of change with one another (transaction and velocity graphs) and the graphs of accumulation with one another (balance and position graphs). While I did not expect fifth graders to arrive at this same level of abstraction, I wanted to determine whether they could make some connection between the rate of change concepts the explored in different contexts.

Towards the end of the interview, I presented students with the following problem. *I want you to think again about the motion graphs. Suppose you were to walk away from the motion detector to the wall, and then back again. Can you figure out a way to model this motion using the Bank Account Interactive Diagram?* All six students were able to use Bank Account to model this motion, however their level of understanding differed.

Some students focused primarily on transferring a motion graph of position versus time into Bank Account, without thinking about where they should create the graph (in the transaction graph or the balance graph) or how the graphs of motion related to the graphs in Bank Account. For example, both James and Carlos formed a picture about the shape the graph should have (see figure C, James's drawing), and then tried to create that shape using the bank account program.

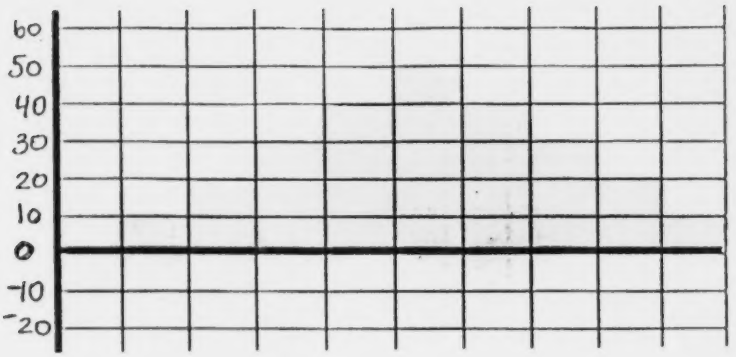

Figure C

James decided he would model the motion in the transaction graph, and proceeded to enter positive transactions of 10, 20, 30, 40, 50 and then 40, 30, 20, and 10 to create the following graph (see figure D)

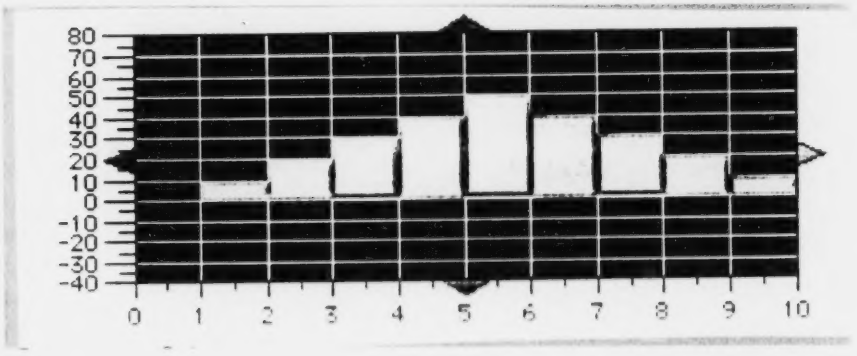

Figure D

E: Ok James, tell me about your graph.

J: Well its walking away, then walking back. The motion detector, the further away you are, it goes up. The numbers (the numbers on the vertical axis of the transaction graph) represent how far away you went. Say you went 10 feet away, it goes like this (points to the first transaction, 10) and then 20 feet away, it goes like this (points to the second transaction of 20. Then when you are walking forward, the distance between you and the motion detector shrinks down more (points to the second half of the graph).

James successfully used the Bank Account Interactive Diagram to model the motion of walking away from and back toward a motion detector. He understood that adding quantities of money was analogous to an increase in position (walking away), and then chose to add less money to model a decrease in position (walking towards). He correctly assigned meaning to the numbers on the vertical axis of the transaction graph, claiming that they represent distance away from the motion detector.

However, James did not make a deeper connection, or rather abstraction, between graphs of motion and the graphs in Bank Account. For example, he did not think about where he should create the position graph modeling the motion, he simply created it in the transaction graph. James did not consider that the transaction graph, which represents changes, is actually analogous to a velocity graph, and that the balance graph, which represents accumulated amounts, is analogous to a position graph. His decision to use the transaction graph to model the motion may have been influenced by the nature of the Interactive Diagram, which only allows users to operate on the transaction graph. To model a position graph in the balance graph, James would first need to enter a series of transactions in the transaction graph that would create the appropriate balance graph.

Unlike James, Carmen chooses to use both graphs in the Bank Account Interactive Diagram to model the motion. Interestingly, she also began by predicting the shape of motion graph, and her prediction was identical to James's. (see figure E)

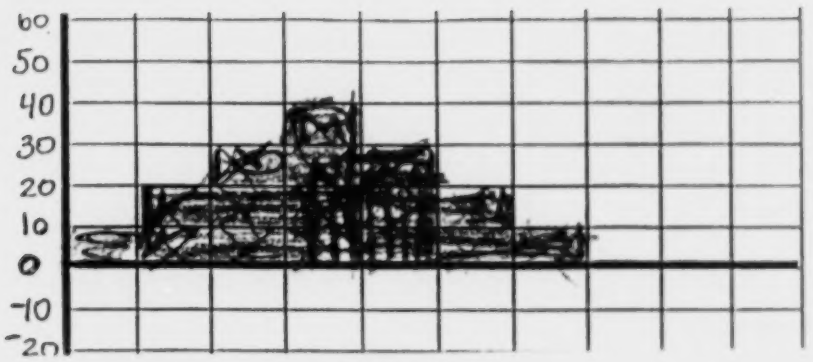

Figure E

As she explained her prediction to me, she commented..

C: You back away and then come back. It would be like.. you are adding money and then taking away money, or maybe adding less money. It shows you are walking backwards, and then right here you come forward. (traces her finger along the graph she drew, following it up and then down)

E: What would be on this axis? (the vertical axis)

C: The time (points to the horizontal axis) and the speed (points to vertical axis) The velocity I mean.

E: Tell me why you think that would be velocity.

C: I think it would be velocity .. but it also shows how far away you are from the motion detector.

E: Which one do you think it would be?

C: On this graph (points to her prediction), I think it would be how far away you are.

I then asked Carmen if she could use Bank Account to model this motion. Instead of creating the graph she predicted (figure D) directly in the transaction graph, Carmen generated a series of transactions so that would create a model of the motion in the balance graph.

C: Right here(pointing the first step up on graph she drew), I added 10, so I'll put in 10. Then you go another day, at 10. Again, again, again. (She adds 10 dollars for three more days) Then you take away 10. You put in negative 10. Then again, again, again, and another time. (She takes away 10 for four more days). Figure F displays the graph Carmen created.

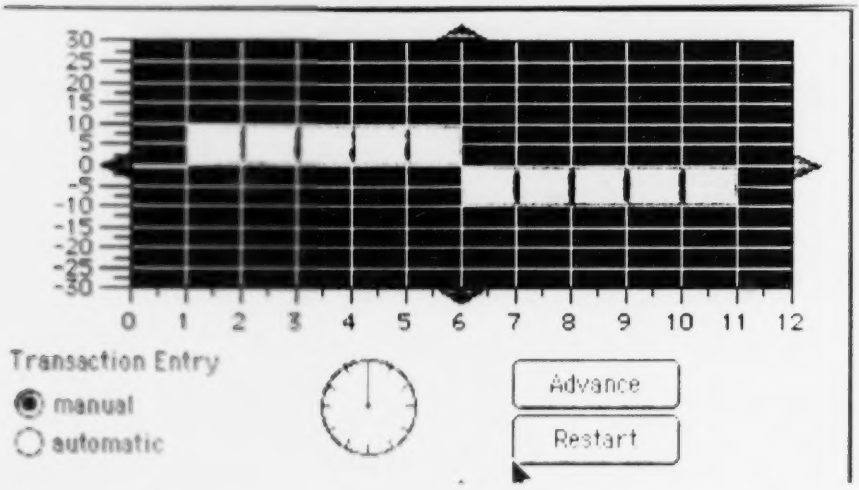

Figure F

E: So what do you think'?

C: The balance is in the transaction. In my prediction.

E: What do you mean?

C: I mean here (points to the graph she predicted on paper) I used the transaction, to show the balance. It should be in the balance, like on here (points to the balance graph on the computer screen) Cause right here (points to the transaction graph on the computer screen) it has to show that you are taking away 10, and then adding 10.

E: How did you know, that in Bank Account, you should add 10 and then take away 10?

C: Because right here (points to her the graph she predicted on paper) it kind of goes up a little but, and then so it shows you that you add 10 and take away 10. ... cause right here, its going down 10 every time.

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While Carmen was not able to fully articulate why she chose to model the position graph of the motion in the balance graph, she clearly has connected representations of change across different contexts. She immediately realized that an increase in position in analogous to adding money, and then wavered between whether a decrease in position entailed subtracting money or adding less money. It is almost as if she was envisioning the position graph in both locations. To model the decrease in position in the transaction graph, she would need to add less money, whereas to represent the decrease in the balance graph she would need to take out money.

Carmen struggled to determine what each graph represented, and where she should model the position graph, which I attribute to her efforts to make some kind of connection between the contexts. She finally concluded that the position graph she predicted belonged in the balance graph, and that the transaction graph should display the addition and subtraction of quantities needed to create the balance graph. I am not at all convinced that Carmen connected the graphs of change on an abstract level, but her responses revealed that she had begun to look for connections, and that she had an intuitive sense about how the graphs might be related.

While Carmen, like other students such as Fatima and Heather, was just beginning to connect representations of change across contexts, Kristina demonstrated a deeper level of abstraction. When I first presented the problem to Kristina, she also sketched a prediction on paper of a position versus time graph that modeled the motion. I then asked her if she could model that motion using Bank Account.

K: Ok, the graph, you're going at a constant pace, right? You want me to make sure that it shows up here (points to the balance graph) or up here (points to the transaction graph)?

E: Where do you think that it should go?

K: Down here (points to the balance graph)

E: Why do you think it should go down there?

K: Well, this (the transaction graph) is like the velocity graph, its saying like how fast you are going. And then this (the balance graph) is like.. I can't remember what it is called. The top graph, with the motion detectors…

E: The position graph

K: Yeah, the position. …

Kristina immediately decided that the transaction graph represented the velocity and that the balance graph should represent position. She developed and further articulated this connection as she proceeded to create the graph.

K: Now, we want to type in here (points to transaction graph), like at a constant pace, right?

E: Right, at a constant pace walking away and then back again.

K: Ok, now we'll put in 10. 1, 2, 3 .. 10 days, we'll do that so it would go at a constant pace up here. (enters a transaction of 10 dollars per day for 10 days) It would go at a constant pace, all the way across. So we go down here, and we say display balance. (see figure G)

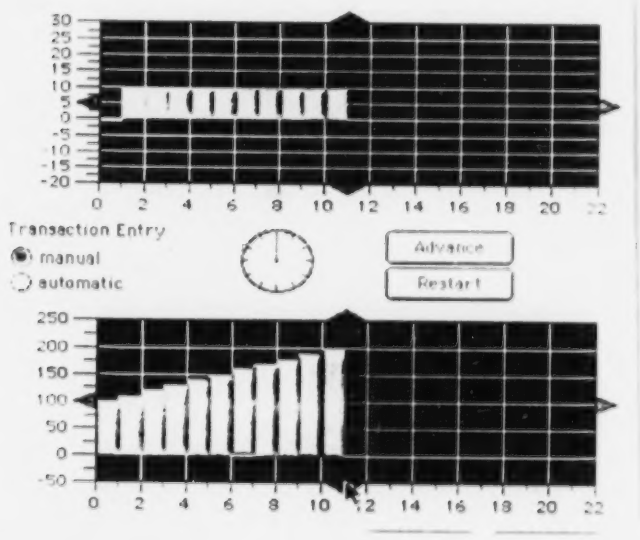

Figure G

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K: It goes up at a constant pace, and then we want it to go down like that again. .... We'll say, 10, negative 10.

E: Tell me why you are putting in negative 10?

K: Because we want it to go up one amount and then down that amount. (Enters a transaction of negative 10 per day for 10 days) Now we are going to go down here (points to balance graph) and it will start to go down again. (traces a line with her finger to show the shape of the graph) Now we go down here and display balance. And it is just what I thought, it goes up and then back down to 0. (see figure H)

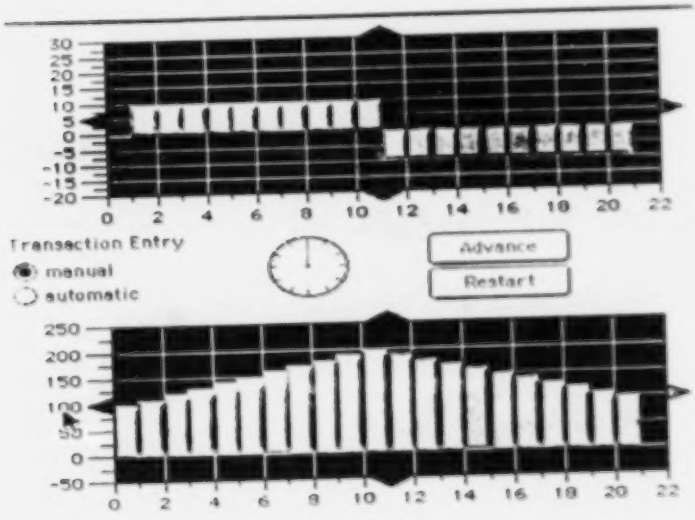

Figure H

E: And can you tell me, how your graph models the motion?

K: This part right here (points to first part of the balance graph, sloping upward, where it is going up like that and it stops right here, that's where you are going back, and then you walk forward.

E: And on the top graph, which you said is like a velocity graph, how did you decide to enter 10, 10 ,10, 10..?

K: Ok, you want to go up the same pace each time, so to say you are going up at the same pace, so you are going to go 10.

E: And how did you know to take out day for all those days?

K: Cause you want to go down at the same pace, go down 10 each time.

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E: I have one more question for you, how did you know right away that the top graph (transaction graph) was like a velocity graph?

K: Well, its because in Bank Account, it shows how much you put in each day, but in the other one (motion detectors) it shows what speed you are going.

Kristina made a definite connection between the meaning of the graphs in the motion context and the meaning of the graphs in Bank Account. She talked about how the transaction graph represented the pace, or velocity, and that a series of constant transactions would model movement at a constant pace. She related "going up" in the balance graph with walking away in a position graph, and "going down" with walking forward, and she knew that if she wanted to walk at a constant pace, she needed to represent that pace in the velocity graph.

As Kristina worked on this problem, she seemed to have both contexts, motion and banking, in mind. She stated that she would "type in here (the transaction graph) .. a constant pace." She knew how to represent movement at a constant pace with graphs of motion, and connected that knowledge with her experiences modeling constant transactions in Bank Account. Kristina also knew that walking away and back again at a constant pace meant producing a graph that goes up and down at the same pace. She said, "Because we want it to go up one amount and then down that amount." She knew what the position graph needed to look like, and knew to enter a constant positive transaction to create the first portion of the graph (walking away). However, I do not think it was immediately obvious to Kristina that she needed to enter negative transactions to model the negative velocity of walking back towards the motion detector. She created the first portion of the transaction graph, then looked at the resultant balance graph and said, "It goes up at a constant pace, and then we want it to go down like that again. .... We'll say, 10, negative 10." I think that Bank Account helped her to make sense of the negative velocity, because she knew she needed the balance graph to go down at a constant pace, which would require a constant negative transaction. Thus, I think Kristina's responses not only

demonstrated that she made a significant connection between graphs of change across contexts, but that trying to make this connection also helped her to better understand the graphs in each individual context.

3. Students were able to use their understanding of rate of change as a lens to interpret and make sense of new representations and situations

For the final question of the interview, I presented each student with the following graph. (see figure I)

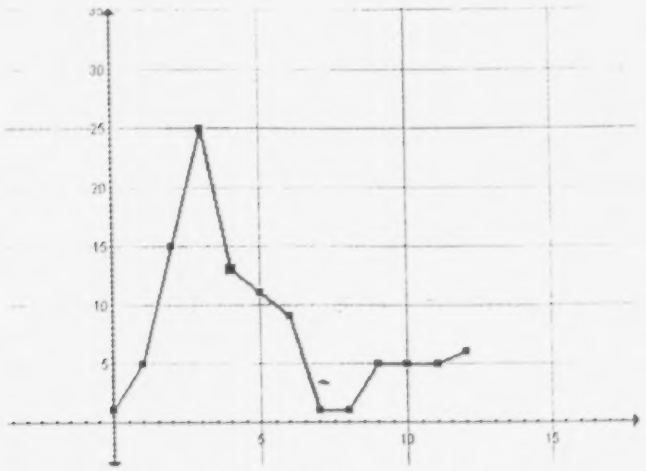

Figure I

I explained that the graph could be a graph modeling motion or banking, and then asked the student to think of some other situation or context that the graph could represent. The purpose of this question was to provide insight into the students' understandings of more general graphical representations of change, representations that were not linked to a motion or banking context. I wanted to determine whether students could draw upon their understanding of rate of change concepts and use this understanding as a lens to "recognize" and generate other situations that contained elements of change. Contexts that I thought students might generate included:

changes in temperature or rainfall over time, changes in value (money value, stock value), and changes in earnings. Interestingly, while five of the six students successfully described a situation that the graph might be modeling, and three of the six students generated multiple examples, none of their examples corresponded to the more standard, familiar contexts of change that I thought they might associate with the graph. Instead, their responses revealed a deep analysis and understanding of the meaning of the graph, creative and flexible thinking, and an ability to draw upon situations of change that are relevant to their own lives.

As the students explained the context that the graph might represent, they attended to salient features of the graph, such as changes in position and slope. For example, James commented that they graph could model changes in car sales.

> E: Suppose you were given the following graph. This could be a graph modeling motion or banking. Can you think of anything else this graph could be modeling?
>
> J: cars sales, you know business
>
> E: Can you explain that for me?
>
> J: The car sales. You were selling no cars (points to the origin of the graph) at first. Then it kept going up and up and up (points to the initial incline of the graph) and then the sales started going down. First it went down at one rate, and then they went down at a different rate and then a different rate (points to the three portions with a negative slope) and then, you were selling no cars (points to the flat portion of the graph near the x axis). And then it went up, and then it stayed at that rate for a while (points to the second flat portion of the graph). And then, it started going up again, you started selling more cars.
>
> E: Ok, now is this graph showing how many cars you were selling, or how fast you were selling those cars?
>
> J: Both, how many and how fast.

James's comments demonstrated a solid understanding of the meaning of this graph of change. He related the ascending and descending portions of the graph to increases and

decreases in car sales, and realized that changes in the steepness of the graph represented changes in the rate of sales. He commented, "First it went down at one rate, and then they went down at a different rate and then a different rate" as he described the various decreases in sales. James also understood the different meanings of the two flat portions in the graph. He explained that the first flat section (essentially at zero) meant no car sales, while the second flat portion (at the level of 5 on the y axis) represented a constant rate of car sales. He stated, "and then it went up, and then it stayed at that rate for a while." James's comment that the graph represented both how many cars were sold and how fast the cars were sold also demonstrated the depth of his understanding. He realized that a single graph could provide information about the amount of sales (through the height of the graph) and the rate of sales (through the steepness of the graph).

James used much of the same language to describe this graph that he used to describe graphs of motion earlier in the interview. When he analyzed graphs of motion, he explained that steep portions of the graph indicated faster movement, less steep portions indicated slower movement, and flat portions meant no movement at all. He also stated that graphs with a steep incline (such as the graph in figure J) represented movement that was both fast and went "far back", which may have helped him understand the his graph of car sales represented both how many were sold and how fast there were sold. I believe James, and other students, drew upon these understandings to help them make sense of this new graph.

Figure J

While James attended closely to each section of the graph, and generated a situation that mirrored the changes in quantity and changes in rate of change, Heather considered the graph more generally, as a representation of change. She then tried to think of as many situations as possible that the graph could model. Her response demonstrated an ability to think flexibly about change, and an understanding that elements of change are integral to many real-life contexts and situations.

E: What else (besides money and banking) could that graph stand for?

H: It could stand for the prices, no the sales tax, the sales tax cause sometimes it goes up, and then comes back down and crashes, and then goes back up.

E: anything else?

H: It could be the different languages present in the United States, because when people move back to their country and stuff. So at one point it would be down here (points to origin of the graph), and then it would go up and back down and then go back up, because people are going to visit, and then switching (motioning with arms people going back and forth to demonstrate people travelling back and forth between the US and other countries)

E: So what would it mean when it was real high?

H: That would mean a lot of different languages spoken. And low would mean not a lot of different languages.

E: OK

H: And maybe it could be male and female. Male would be coming up at one point and going down at one point and female going up at one point and coming down at one point.

E: How could this graph be the numbers of males and females?

H: You would have to have two graphs, the males going up and down and then on another graph the females going up and down. And, it could be how many people watch TV, or the money than is happening from the shows, or the advertisements.

E: Pick one of those and tell me about it.

H: Ok, how many people have a TV would be sometimes it shoots up and then it goes down, because like their parents are mad at them. And if it is the money from the TV, it's because people watch it then the money goes up because people would watch it a lot. And sometimes it goes down, the money goes down, because people don't watch it a lot.

While Heather did not elaborate on any of her examples, and did not articulate what specific portions of the graph might mean, she clearly understood that the graph represented changes over time. She seemed to have a strong sense of quantities that inherently fluctuate over time, such as the amount of people watching television or the numbers of males of females within a certain population. Heather only stopped generating possible contexts for the graph because I needed to move on to the next question of the interview; I sensed that she could have continued thinking of "stories" for the graph for quite a while.

Several of the students, including Kristina and Fatima, drew upon their personal lived experiences as they assigned meaning to the graph. For example, Kristina stated that the graph could model changes in the amount of homework she received from day to day.

E: This is a graph that could be used to show motion, or money in your bank account. I want you to think of something else that this graph might represent.

K: Homework. One day you have 5 papers of homework, the next day you have 10, the next day, 10 more. You keep getting more and more.

E: And so on this day? (pointing to the highest portion of the graph)

K: You got a lot, about 25 papers. And then the next day, you are taking off paper, it would be about 12. You are taking out 12, 13, well, its like you're getting less homework, and then you're getting less, less, less. (pointing to the portions of the graph with a negative slope)

As Kristina assigned meaning to this graph, she thought about her own life, and situations involving change that form part of her daily lived experiences. Her comments not only demonstrated that she understood the graph as a graph of change (change in the amount of homework she received from day to day), but that she understood the graph as something that could relate to her own life. Many teachers strive to help students make connections between their lives and the mathematics they study in school. Kristina spontaneously made such a connection on her own.

Fatima also drew upon her personal experiences as she described a situation that the graph might represent. When I posed the question to Fatima, she traced her finger along the graph and described how each section of the graph depicted an increase or decrease in quantity.

> F: Le vas a sumar, luego le quitas, luego le vas quitando, quitando, Le quitas hasta abajo. Luego sumas, luego aqui vas sumandole, y aqui le sumas un poquito más, más de lo que sumaste aqui.
>
> *Translation*
>
> *F: You are going to add, then take away. Then I am going to add this and then I am taking away some. And here you are adding some, and then you add a little bit more.*

I then asked Fatima if she could think of a situation that the graph might be modeling. She immediately started explaining to me how the graph could represent the process of measuring and altering fabric while sewing á dress.

> F: Cuando estás haciendo un vestido o algo, y haces medidas por un lado o por el otro. Por ejemplo, cuando van a arreglar un vestido, dicen "quiero que me lo agarren aqui, que me lo suelten acá",y asi.
>
> E: Enseñame como sería en la gráfica, este ejemplo del vestido?
>
> F: Por ejemplo, si es un vestido que le queda apretada de las piernas o asi, quiero que me agarran las pinzas que se abra más. Entonces, a abrirle más van a sumar aqui lo que es lo que van a abrir. Entonces acá abajo quiero que me lo quiten más. Entonces, le vas a

quitar lo que vas a quitar. Quiero que me le quiten aqui, y luego aqui asi me le dejan. Luego, que me le suelten más, le deja igual y acá le abre un poquito más.

Translation

F: When you are making a dress or something, and you make measurements, on one side or the other. For example, when they are going to alter a dress, they say "I want you to take it in here and let it out here."

E: Show how that would be on the graph, your example with the dress?

F: For example, if it is a dress that is tight in the legs or something, you get the _____ that will open it up more. Then, to open it up you are going to add on the amount that you are going to open it (points to the initial ascending portion of the graph). Then there below, I want you to take some in, then you are going to take away what it is that you need to take in (points to the descending portion of the graph). Then, here take it in for me, here leave it just like it is (points to the flat portion of the graph near the x axis). Then let it out for me, then leave it the same, then let it out just a little bit here.(points to the final sections of the graph: a steep incline, a flat portion, and an incline that is not as steep)

While Fatima's example is difficult to follow, and arguably not a valid context for the graph, I believe her example deserves mention and consideration. After the interview, I asked Fatima how she thought of this sewing example. She explained that her mother is a seamstress, and frequently has clients visit her home so that she can alter their clothing. Fatima often helps her mother as she marks which portions of the garment need to be taken in and which portions need to be let out. She said that the dresses are always changing, getting bigger in some places and then smaller in others.

Fatima's response demonstrated that she could correctly interpret a graph of change. She identified which portions of the graph represented increases and decreases in quantity, she distinguished between increases of different magnitudes, and explained where the graph represented a quantity that stayed the same. However, while the context she chose does include quantities that change (fabric that is let out or taken in), applying that context to the graph

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becomes problematic. I am unsure what the axes would represent, and unfortunately I did not probe Fatima further to find out what meaning she would assign them.

Yet Fatima's story of altering a dress clearly demonstrated to me that she did understand the graph, and that she was intent upon connecting the graph to a situation that made sense to her. As I did not ask either student, Kristina or Fatima, to think of a situation from their own life that the graph could represent, their ability to make a personal connection with the graph is all the more meaningful. Their responses indicate the power of rate of change concepts to not only engage students in the study of significant mathematics, but to encourage students to relate the mathematics to their own lives and experiences.

4. . Students were able to use the banking context to conjecture and experiment to build a system of consistent rules about positive and negative numbers

While I originally did not intend to investigate how Bank Account might help students to understand operations with positive and negative numbers, as the students worked in Bank Account during the after school program, I was struck by their facility with negative numbers. Students knew that if they subtracted 100 dollars from a bank account with a balance of 50 dollars, they would end up "50 dollars in the hole", with negative 50 dollars. They could then continue to subtract money from this negative balance, and calculate the result. Students even came up with generalizations and rules about working with negative numbers, such as "when you subtract money from a negative, you just go more in the hole. It' still a negative." Since addition and subtraction with negative numbers is not a formal part of the elementary curriculum at Longview, our work adding and subtracting money in Bank Account was some students' first experience with negative numbers. It seemed that Bank Account provided a very powerful model for students, one that prompted them to think about a system of rules for positive and

negative number operations, and one that provided immediate feedback to support them in constructing those rules. To further explore the extent of students' understandings about negative numbers and the system of rules that govern their behavior, I decided to include some positive and negative number problems at the end of the interview.

I presented each student with a series of calculations involving positive and negative numbers. (see appendix D) The problems began with easy calculations with two positive numbers, such as 12 – 5 and 12 – 20. The problems became progressively more difficult, involving adding two negative numbers (-15 + -2) , subtracting a positive from a negative (-20 – 10), and subtracting a negative from a negative (-15 - -10). I first asked students to solve all of the problems, and then to explain some of their solutions to me. I encouraged the students to prove their answers to me, either by explaining their reasoning or by presenting a situation or "story" to demonstrate how they thought about the problem.

Overall, students were quite successful solving these positive and negative number calculations. I presented each student with 17 problems, and all students answered at least 15 of the 17 questions problems correctly. One student, James, correctly answered all of the questions. The problems that several students missed were some of the most difficult (e.g. –15 - -10 and –15 - -20). (note: Fatima did not complete this portion of the clinical interview because we ran out of time.)

Most of the students spontaneously used Bank Account like situations involving transactions and money to explain their solutions and reasoning to me. For example, to justify his answer to –15 + 2, Carlos presented the following argument. He clearly understood that adding a positive amount makes a number larger, even when the number he started with was negative.

E: What about this one, -15 + 2?

C: I got negative 13. Cause you are adding 2, it's not negatives anymore. If you are adding 2 to negative 15, then the number is going up because you are adding.

E: How do you know?

C: It's like, you are in the hole 15 dollars, and you get 2 dollars for an allowance, so how much money are you in the hole by now? 13.

James also used situations involving money to explain many of his answers. Like Carlos, the context of owing or borrowing money seemed to make sense to him, and he used this context to explain negative numbers. What follows is our discussion about his answer to –20 – 10.

E: Ok, so how did you get –30?

J: Negative 20 minus 10 more. Its like adding more to what you already have, and then adding a minus sign to it. … Say you owe your mom 20, and then you have to borrow 10 more, you still owe your mom, you owe her 30 dollars.

Like Carlos and James, Heather frequently presented examples involving money as she explained her answers to me. Even as she solved the problems on her own, she often talked through the problem out loud, making continual references to adding and subtracting amounts of money. For example, as she first solved the problem –15 + -2, she made the following comments .

H: (trying to solve –15 + -2) Ok, if you have –15 dollars in the bank account, and you add minus 2.. How would you add minus 2? Would you take out? Would that be like taking out 2 dollars? So you would go down to minus 17. Yeah, because you have minus 15 and how would you take out 2, and in this case you are adding. But just say you are taking out. So you had to take out 2 dollars, so it would be like adding negative 2 dollars to this (the negative 15). So then it would be –17.

While Carlos and James relied on a money context primarily to explain their reasoning, Heather drew upon her experiences with Bank Account as she reasoned through the problem. She used the context of banking to help her understand how to add two negative numbers; she

concluded that adding a negative number is analogous to taking money out of the bank. As she continued to solve similar problems, such as 10 + -5 and 10 + -30, she relied on this connection. For example, when Heather solved 10 + -5, she said, " Ok, 10 + -5, If you are subtracting 5, well you are adding minus 5 for that would be like taking away 5. So it (the answer) would be 5." Thinking about money and banking allowed Heather to define adding a negative number as subtracting a positive number. But once she made this connection she was able to use it to solve other problems, without having to explain every problem in terms of money.

For some students, such as Heather and Carlos, thinking about Bank Account helped them to decide which of a few possible answers was the correct answer. When they limited their thinking to the numbers themselves, they could not tell whether an answer was 5 or –5, or 8 or –8. But when they thought about the problem in terms of money in their bank account, the correct answer seemed obvious to them. For example, Heather initially arrived at an incorrect solution to the problem 12 –20.

E: Okay, so how did you do 12 – 20?

H: I just did 20 – 12, and that is 8, so its 8.

E: How did you know you could do 20 – 12?

H: Cause its just switching it around like you do in other problems.

At this point, I decided not to challenge Heather's reasoning. However, after she successfully solved several other problems, I asked her to think about 12 – 20 again. What follows is her solution to –20 – 10, and then her reconsideration of the12 – 20.

E: What about this one, -20 – 10?

H: Well, that was confusing at first. I was saying 10 cause I was thinking of adding. Well its like just adding those, just adding cause you're adding 10 to a 20, except this time you're subtracting it. If you're subtracting 10 from 20, It would be 10. But in negative numbers, it would be 30, minus 30.

E: How did you know that? Show me how you know that it is true.

H: Say you had -20 dollars, and then you took out 10 more, so that would be –30.

E: I am curious if you can tell me a story for 12 – 20.

H: You have 20 dollars in your bank and you take out 12, how much do you have left? 8

E: And how do I know that I start with 20 and I don't start with 12?

H: Because if you took out 20, that's not possible, its like taking 5 out of 4. You can't take 5 out of 4, you can't do that.

E: Really, what happens if I have 4 dollars in my bank account and I take out 5 dollars?

H: minus 1 dollar

E: So what happens if I have 12 dollars in my account and I take out 20?

H: That would be minus 8 dollars. Hmm, wouldn't that be .. yeah, that would be –8, so it (the answer to 12-20) can be –8 or 8 dollars.

E: So that problem has 2 answers?

H: It could be either.

E: What would it be if it was your bank?

H: It would be a negative, cause you would have to have –8 dollars.

I found this interaction with Heather quite intriguigng. While she had successfully solved other problems and arrived at negative answers, she seemed reluctant to use negatives to solve 12 –20. In fact, she asserted that it was impossible to take 20 away from 12, because she needed to have 20 to take away 20. Interestingly, subtracting from negative numbers did not seem to bother her at all. It she started with –20, she had no problems taking 10 away (e.g. she said –20 – 10 = -30), but taking 20 away from 12 was problematic for her. However, when I introduced the context of banking by stating, "Really, what happens if I have 4 dollars in my bank account and I take out 5 dollars?", Heather did not hesitate at all to subtract 5 from 4 and arrive at an answer of negative 1. While I cannot be sure what caused her reluctance to subtract a larger positive number from a smaller positive number, I suspect that her prior schooling experiences might have been a factor. (e.g. Children are taught that they cannot subtract larger numbers from smaller ones, they need to borrow) Regardless of the cause, thinking about subtraction in the

context of banking helped Heather to make sense of how she could solve 12 – 20, and how she could decide which answer, 8 or negative 8, was correct.

Kristina was the only student who did not use a context of money or banking to help her make sense of the problems. Instead, she thought of negatives as numbers that represent the difference between what the amount she had and the amount she needed. For example, to explain her solution for –20 + 50, Kristina told the following story.

> E: What about –20 + 50? How did you solve that one?
>
> K: There is a grocery store, it had 40 workers, but they needed 60, so they didn't have 20, negative 20. SO they got 50 more workers. So the 50 plus the 20 they needed. So you had the 20 that they needed, you add 50, that is gone (points to the –20) so you need 0. And then you have 30 left over. So the answer is 30.

Kristina used the same line of reasoning to explain almost all of the problems that she solved. She never just started with a negative amount (e.g. you have –20 dollars), instead she presented a situation that involved a discrepancy in amounts (e.g. you had 40 workers but you needed 60) and generated the negative number from that discrepancy. Kristina also consistently used the strategy of breaking apart the numbers to arrive at 0 and then dealing with the amount she still had left. For example, as she continued to explain her reasoning about how she solved the problem –20 + 50, she commented:

> K: Ok, so it was –20, and if it was –20, it would go up to 0. (writes –20 + 20 = 0). So we'll do 50 minus 20 equals 30. (writes 50 – 20 = 30) That's 30 you have left.
>
> E: Ok, first you did –20 up to 0, then you did 50 – 20. Why did you do 50 – 20?
>
> K: Take away the 20 from the 50 that you used to get to zero, so there is 30 left.

Kristina's goal was to "get rid of" the negative, and she used whatever portion of the positive number she needed to arrive at zero. Kristina also used the strategy of arriving at zero when she solved problems like 10 + -15, where it was impossible to reach zero. She knew that adding -15 plus 15 would reach zero, but realized that in this problem, so did not have positive 15, she only

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had positive 10. She said, "We need to do negative 15 plus15 equals zero. Then you would get to zero. There is only 10 though. So we are going to do 15 minus 10 equals 5, so we will only be up to negative 5. Because we didn't have that whole 15 to get to zero."

Kristina answered all but one of the questions correctly, and consistently made use of these two strategies: referring to negatives as discrepancies between amounts, and breaking apart positive numbers with the goal of arriving at zero. Kristina's responses demonstrate that some children have alternative ways of making sense of positive and negative number problems, ways that do not rely on the contexts of money and banking.

As the students proceeded through the series of problems, they often generalized a strategy for solving a particular problem and explained how that strategy would work with other similar numbers. The students were usually careful to state under what conditions a strategy would work (e.g. adding 2 negative numbers) and under what conditions a strategy would not work (e.g. adding a positive to a negative). For example, in response to my question about how she knew that the answer to 10 + -30 would be –20, Heather made the following generalizations about adding positive and negative integers.

> H: Well, because if you are adding 2 positive numbers it would be a positive number, and if you are adding 2 negative numbers, it would be a negative number, and if you are adding a positive and a negative it can be either. If the negative number is bigger than the positive number, than it will be negative, and if the negative number is smaller than the positive number then it will be positive.

Other students also spontaneously generated their own "rules" that should govern calculations with positive and negative numbers. What follows are comments from Carmen and Carlos.

> Carmen: (for –20 – 10) Well, when you subtract you really add. Because you can't subtract anything from a negative. Well you can subtract from a negative. But it won't go closer to the positive like this one (-20 + 5) it will go even farther from a positive.

Carlos: (for –15 - -10) Then I realized that when you take away some negatives, it really means that you adding something.

While some of the "rules" these students generated are not entirely correct and do not consider all cases, they still exemplify the kind of powerful mathematical thinking that we want to foster in upper elementary and middle school students. They exemplify students' desire to construct a system of consistent rules (though not a system of rules about rate of change). For example, Heather thought through each possible combination of adding 2 integers, and reasoned about how negative signs and the relative size of the positive and negative numbers would affect the answer. And even though Carmen failed to consider the case of subtracting a negative from a negative when she stated that subtracting from a negative will "not go closer to the positive", her rule still represents an important step towards more accurate generalizations.

Discussion and Analysis

The results of this study suggest that early introduction to rate of change concepts is valuable; elementary students are clearly capable of thinking about and understanding concepts related to rate of change. Stroup (1995) discussed features of activities that allow young students to explore concepts of rate of change; he stressed that "learners need to be encouraged to explore the ways in which graphs represent not just where things are, but how fast they are changing." (p. 16) As students analyze and construct graphical representations of change, they need to deal with notions of "how fast" as well as notions of "how far." The students in my study explored and demonstrated understanding of both of these ideas. In addition, they not only discussed these ideas in relation to graphs of motion, but in relation to graphs of accumulated quantities in Bank Account and other graphs of change (e.g. the graph presented in the clinical interview, see figure 1)

The results of this study demonstrated that elementary students, in an urban, predominantly minority school, can develop a basic understanding of different representations of change and the connections between those representations within a single context. As students explored graphs of how much and how far (position graphs and balance graph) and how fast (velocity and transactions graph) they not only interpreted each graph individually, but made connections between one graph and another. Students analyzed balance graphs for the series of transactions that produced them, and literally "saw" transactions in the balance graphs. They spoke of balance graphs increasing by constant and changing amounts, and of balance graph increasing at varying speeds. In addition, students realized that a series of constant transactions would produce a "staircase" graph, a graph that either increased or decreased by the same interval each day. A changing rate of change, indicated by changes in the transaction graph, was more challenging for students to interpret. However, they all realized that a changing rate of change would not produce a "straight staircase graph" that changed by a constant amount.

Although students made similar connections between position and velocity graphs while using the motion detectors, the context and arrangement of the Bank Account Interactive Diagram seemed to better facilitate this understanding. Bank Account helped students to make sense of the relationship between graphs of change and graphs of accumulated quantities. As Kristina stated towards the end of her interview, " This one (Bank Account) helped me to learn about the other one (the motion graphs). The other one was kind of confusing. And with this (Bank Account) you just type in a number, and it shows up. You can look at each graph step by step and see what happens." As students used Bank Account, they frequently referred to the information that each graph presented. As Heather described the graphs created by adding 50 dollars per day to her bank account for seven days, she commented, "This one (the transaction

graph) is adding 50 each day and it goes across the same. But this one (balance graph), it goes up and so it shows that you are adding 50 each day too."

I did not observe similar comments as the students interacted with the graphs of motion. Another factor that might have influenced this lack of connection between the position and velocity graphs was the "messiness" of the velocity graphs created by the computer. While students knew that a constant velocity should appear as a straight horizontal line on the computer, the computer rarely produced such a graph. Instead, the velocity graphs of movement at a constant pace appeared as squiggly lines that often jumped both above and below the x axis. Many students became frustrated by their perceived lack of control over the velocity graphs, and simply stopped paying attention to the information they displayed. In contrast, students had complete control over the transaction graphs in Bank Account. They knew that adding money at a constant rate should appear as a horizontal line, and it did.

I sense that returning to graphs of motion after working in Bank Account might be beneficial. In the future, I would like to explore questions that ask students to create position and velocity graphs for movements that involve negative velocities. (Since velocity is directional, any movement walking back towards the motion detector displays a negative velocity). Such questions would help determine whether or not students draw upon their experiences in Bank Account as they decide how to represent the velocity of movement that is decreasing in position.

This study also suggests that exploring rate of change across a variety of contexts and encouraging students to make connections between those contexts, benefits students. Students were capable of "seeing the like in the unlike." (Confrey, 2000) As students attempted to model a motion in the Bank Account Interactive Diagram, they drew upon their understandings of both motion and banking contexts. They relied upon their experiences walking at a constant pace in front of a motion detector, and their knowledge of what a graph of that motion looks like. In

addition, they used their understanding of constant transaction graphs, and how constant transactions affect the balance graph. Some students, such as Kristina, began to directly connect the representations of change across the two contexts. Others, like Carmen and Fatima, made a more intuitive connection, realizing that they needed to enter a series of constant transactions to represent movement at a constant pace, but not yet able to articulate why, or how the graphs (velocity graphs and transaction graphs) related to one another. While I do not believe that students made this connection on their own, prior to the interview, their responses demonstrate that given the opportunity, students are capable of making some level of connection between representations of change from different contexts.

Exploring change across contexts also helped students develop the sense that change is a more general phenomenon, one that is evident in many situations, even situations that are part of their own lives. Students were able to use their understanding of rate of change as a lens to interpret new situations. Once students understood the concepts of change in some contexts, such as banking and motion, they were able to recognize elements of change in many other contexts. In fact, the elementary students in this study generated a total of 11 situations that a qualitative graph of change might represent, significantly more that a group of high school students who were presented with the same question. (Wilhelm & Confrey, in progress) Not only could they think of situations that involved change, but they used those situations to accurately interpret the graphical representation. In many ways, the students' responses to this portion of the clinical interview demonstrated their ability to think more abstractly and generalize their understanding of representations of change, an ability which is quite impressive given the students are only in fifth grade.

Exploring rate of change concepts in Bank Account also seemed to facilitate students' understanding of the system of rules that govern operations with positive and negative integers.

Increasing students' ability to add and subtract with positive and negative numbers was not a topic of focus during the after school program, nor an anticipated result of the study. However, the students' responses during the interview demonstrated that they found Bank Account very useful. When they thought about the context of banking, passing from positive to negative numbers made sense to the students. As Carlos stated when he explained his solution to the problem 12- 20:

> C: At first I was thinking, what miss? You can't do it. But then I realized that you can use negative numbers. Like money, like you are in the whole. . .... Because you can't do it this way unless you do negative numbers. ... Because it would just get you to zero, that's all, and you couldn't really subtract 12 – 20, unless you go to negatives."

The context of banking also helped students make sense of problems that involved adding to or subtracting from negative numbers. They related adding positive amounts to depositing money back into a bank account, which would bring a negative balance closer to zero. Likewise, they related subtracting positive amounts to withdrawing money from the bank, making a negative balance even more negative. The students' ability to draw upon their experiences with Bank Account to help them solve seemingly unrelated problems demonstrates that they viewed the banking context as a problem solving tool. Students used this tool to make sense of and solve problems that most of them had never seen before, and the were quite successful.

The results of this project also illustrate that technology plays a crucial role in helping students develop an understanding of rate of change concepts. The students relied on the feedback the technology provided, using the computer graphs generated by Bank Account and the motion detectors to think hard about the relationships between representations of change. The engaged in a continual process of predicting, testing and refining their ideas about the meaning of the graphs, a process which was enabled by the technology.

Conclusions

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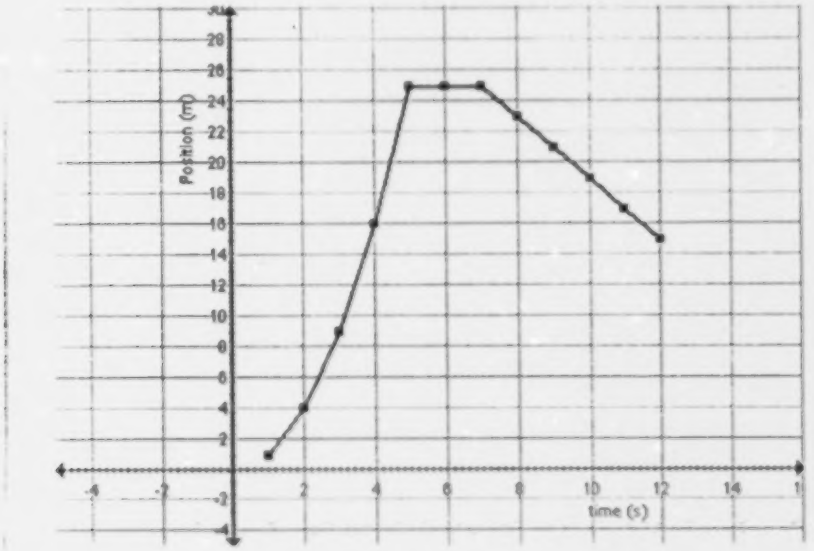
APPENDIX C

Motion Detector – Bank Account Assessment

1. Here is a position versus time graph that one of your classmates created. Explain how the person is moving in front of the motion detector to create this graph.



2. Next your classmate walks at a constant pace away from the motion detector.
- Predict what the position versus time graph would look like.
- Can you model this person's motion?
- Test and check with the motion detector.

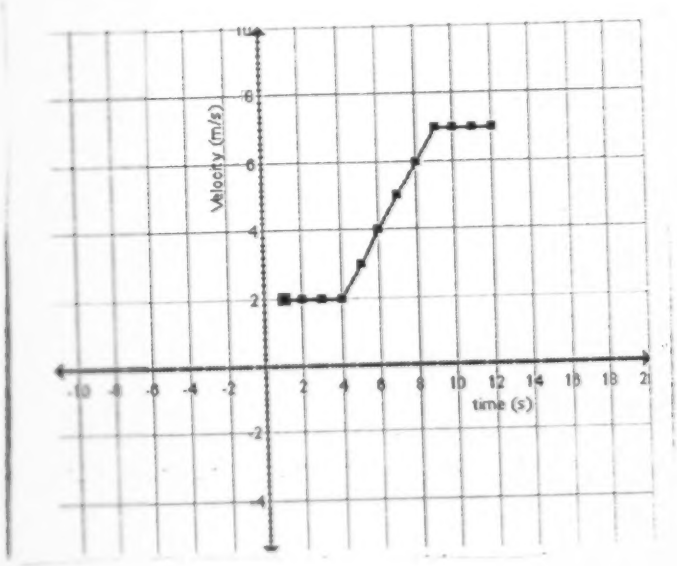
3. Predict what the velocity versus time graph would look like for question 2. Test and check with the motion detector.

4. Next your classmate is running at a constant pace away from the motion detector.
- Predict what the position versus time graph would look like.
- Can you model this person's motion?
- Test and check with the motion detector.

5. Predict what the velocity versus time graph would look like for question 4. Test and check with the motion detector.

6. Compare your results from tasks 2-6. How are the graphs the same and how are they different?

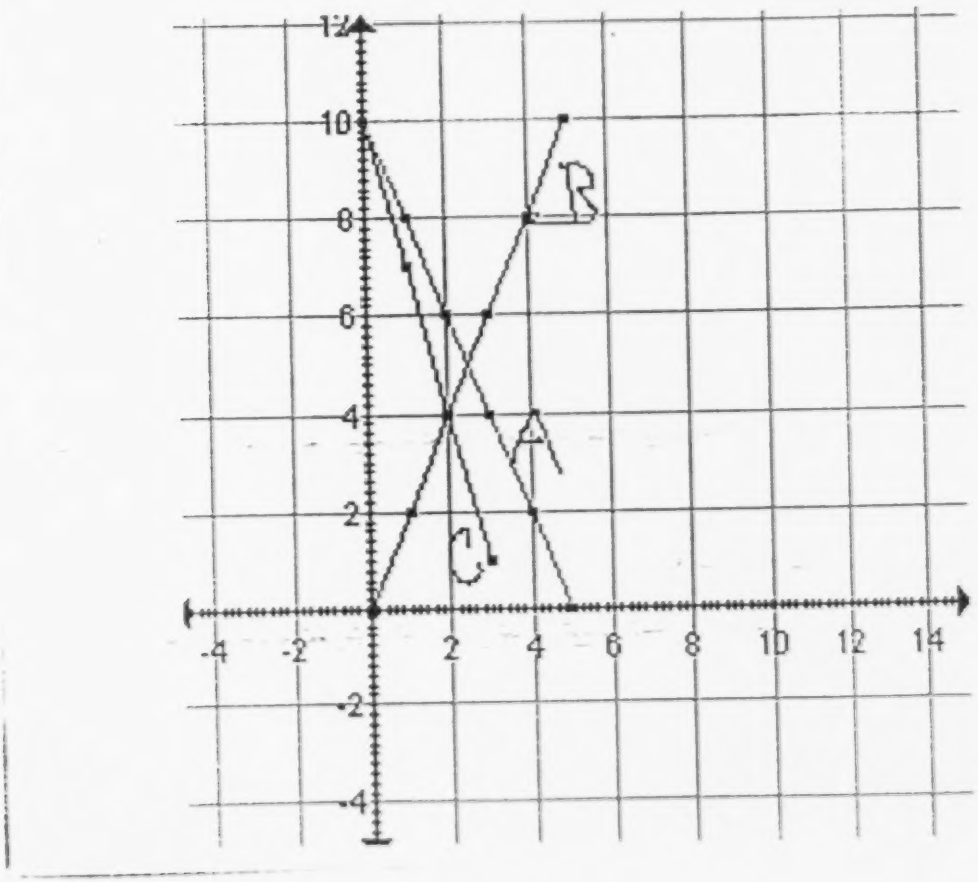
7. Here is a velocity versus time graph that one of your classmates created when they were moving in front of the motion detector. Explain how this person was moving to create this graph.



8. Can you draw the position versus time graph that corresponds to this velocity versus time graph?

9. The following is a graph of position versus time for three people, A, B, and C.

- What positions do each of them begin with and what positions do each of them end with? (Where do they start and where do they end up?)
- Who travels the fastest?



BANK ACCOUNT

TASK 1

1. Suppose you were to open a bank account with an initial amount of $100. The next day, and for each consecutive day a deposit is made of $50. This continues for seven days.

a. Make a prediction of what the transaction and balance graphs will look like.

b. Why is there a bar on the bottom graph at the very beginning?

c. How does the top graph relate to the bottom graph?

d. Will there ever be $400 in the account? If so, when?

NOW STUDENT SHOULD CARRY OUT TASK 1 ON THE ID.

e. Make a model of your savings over this period of time using the Bank Account Interactive Diagram (called "Your ATM").

f. Explain any discrepancies between your predictions and the model on the ID.

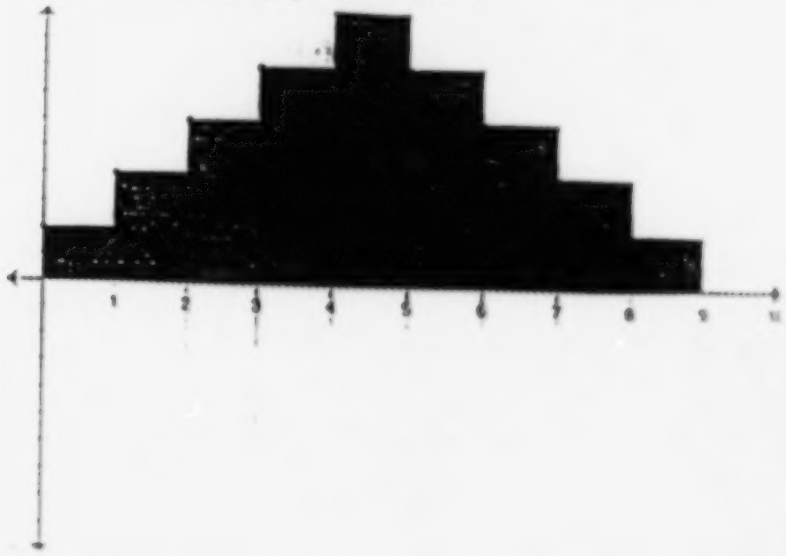
g. What do you notice about the shape of the second graph? How do you explain this?

h. If the amount deposited were twice as big, how would it affect this graph?

i. Repeat (h) with half as big.

TASK III

3. Show student a pre-drawn transaction and ask him/her to predict what the balance graph will look like.



NOW STUDENT SHOULD CARRY OUT TASK 3 ON THE ID.

a. What would happen to the balance graph if we were to double the length of all the bars?

b. What would happen to the balance graph if we were to halve the length of all the bars?

c. What would happen to the balance graph if we were to increase each transaction to $50?

Task III

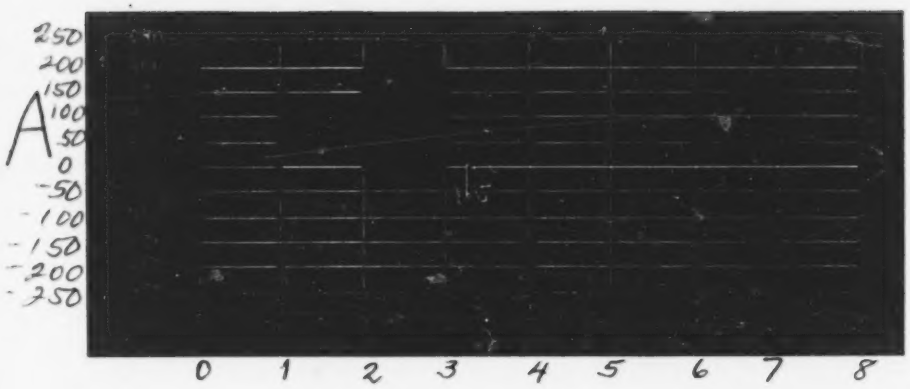
Suppose we have the following balance graphs, A, B, and C.
Compare A, B, and C. How are they alike? How are they different?
Tell me about the daily transaction graphs that produced them.

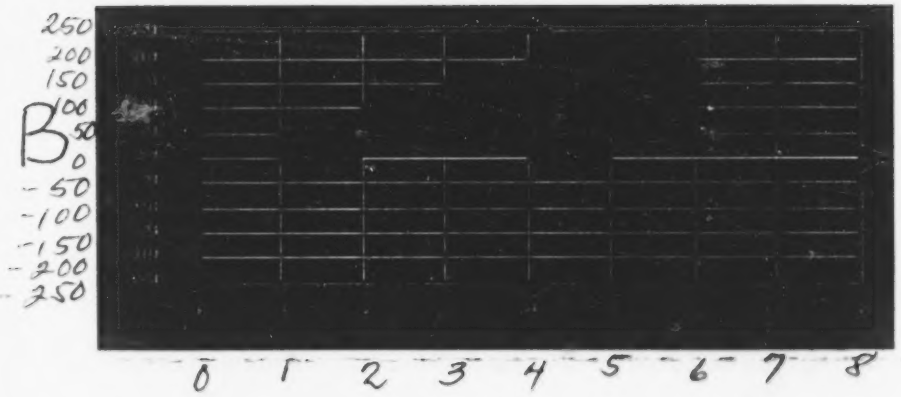
Look at the second set of graphs, D and E. How are they alike? How are they different?
Tell me about the daily transaction graphs that produced them.

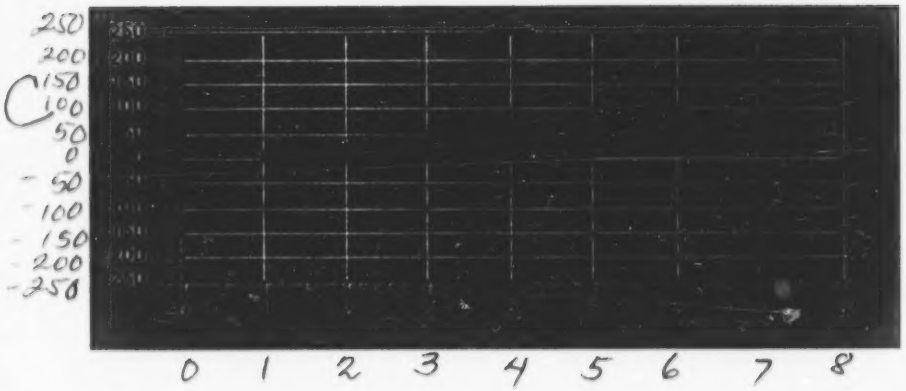
Now look at the last balance graph, graph F. Explain what is happening in graph F. Tell me about the daily transaction graph that produced graph F.

Task IV

Which is easier for you to do:
Predict a balance graph from a transaction graph or predict a transaction graph from a balance graph?
Could you explain to someone else in words how to do this?







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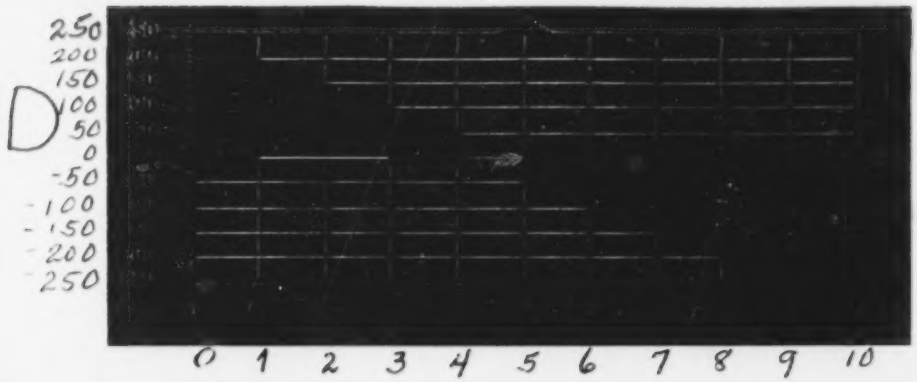

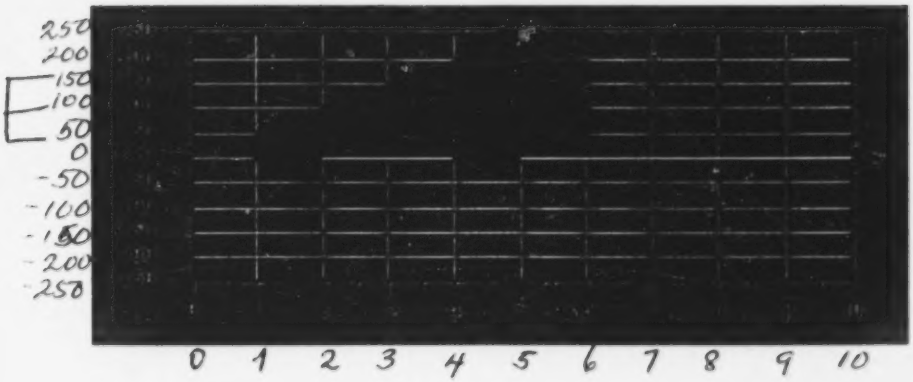

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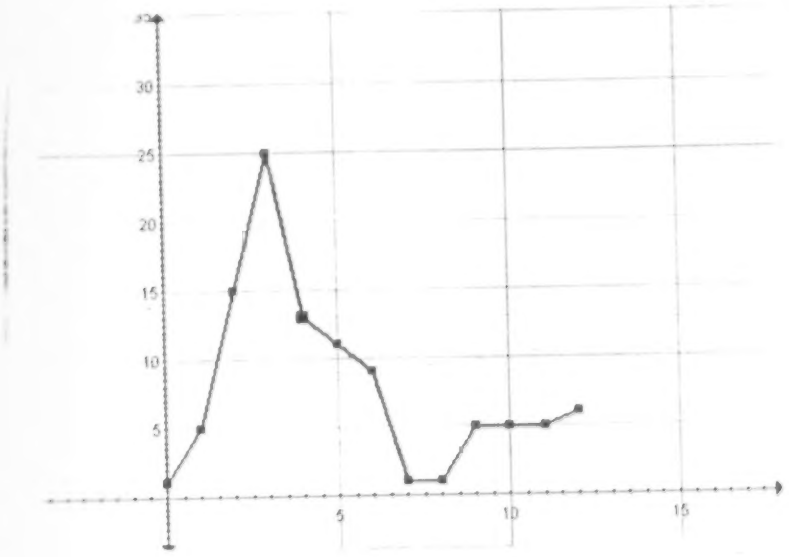
10. *Now let's* think once again about the motion graphs. Suppose you were to walk from the motion detector to the wall and back again. Can you figure out a way to model this motion using the Bank Account Interactive Diagram?

11. Suppose you were given the following graph.

This could be a graph modeling motion or this could be a graph modeling banking. Can you think of anything else that this graph could be modeling?



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$12 - 5 =$

$12 - 15 =$

$12 - 20 =$

$-20 + 20 =$

$-20 + 5 =$

$-20 + 50 =$

$-20 - 10 =$

$-20 - 25 =$

$-15 + -2 =$

$-15 + -20 =$

$-15 + 2 =$

$-15 + 20 =$

$10 + -5 =$

$10 + - 15 =$

$10 + - 30 =$

$-15 - -10 =$

$-15 - -20 =$

$-15 - -3 =$